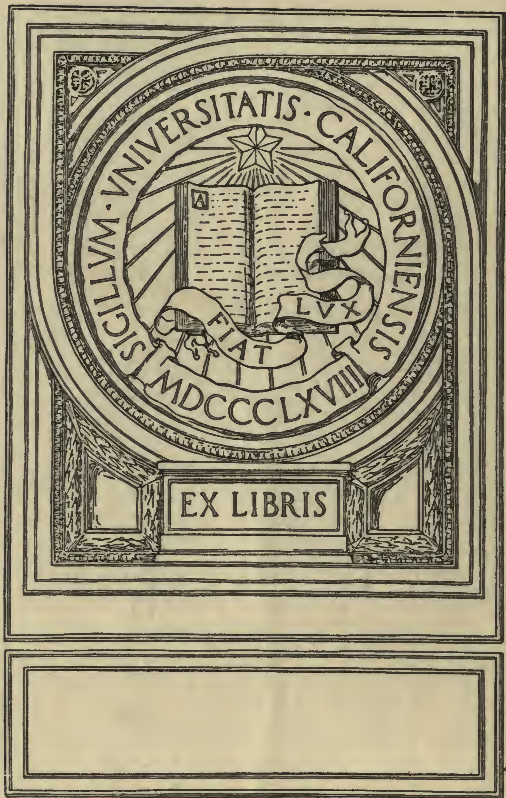
ANSWERS

MILNE'S
SECOND COURSE
IN ALGEBRA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Leon Bauer

SECOND COURSE IN ALGEBRA

E. P. 2

PREFACE

This book is intended to follow Milne's "First Year Algebra," or its equivalent, to provide for the division of Elementary Algebra into two courses.

The general plan and scope of the book have been determined by the recommendations of leading mathematical associations throughout the country and by a careful study of courses of study in many states and cities, including the requirements of the principal colleges and of the College Entrance Board.

As the second course in algebra is usually taught for a half year in the third year of high school, and the first year student is rarely able to retain all that he has learned of algebra when a year intervenes between the courses, the book begins with a thorough review of first year algebra. In the treatment of all review topics, the details of development and explanation are omitted; but the essentials are given, including the restatement of all important laws, principles, and rules. By referring the student to the Glossary for review definitions, the massing of definitions at the beginning of chapters is avoided. The applications in the review are new and somewhat more difficult than those in the "First Year Algebra," to provide for the exercise of increasing mathematical power on the part of the student.

In the chapters containing the requirements of the second course, the new principles are most carefully developed and the explanations are full and clear. These chapters are followed by a general review and by supplementary subjects for optional study.

Each topic is accompanied by a large number of exercises for practice. They provide sufficient work for classes desiring to devote a whole year to the subject. For a half year's work, every alternate exercise may be omitted, at the discretion of the teacher.

Equations and problems are especially emphasized. The problems are based on interesting facts gathered from a variety of sources, including physics, geometry, and business. A few traditional problems are included for the purpose of familiarizing the pupil with them in case they appear on examination papers as well as for their disciplinary value. The formulæ and applied problems are easily within the comprehension of the students for whom they are intended.

Functionality has received brief but sufficient attention in the chapters on Graphic Solutions, where its utility is apparent.

Publisher's Note. — As Dr. Milne did not live to finish the manuscript of the "Second Course in Algebra," the completion of the work was intrusted to his assistant, Mr. Charles R. McKenzie, who for many years was associated with Dr. Milne in the writing of his mathematical works, and who is intimately acquainted with his methods and ideals. To Mr. McKenzie's valuable experience gained through this close association is due the successful completion of the book in entire accord with the author's plans.

CONTENTS

SIGNS AND SYMBOLS

$+$, sign of addition, read '*plus*' or '*increased by*' } also signs of quality.
$-$, sign of subtraction, read '*minus*' or '*diminished by*' } also signs of quality.

$\pm$, ambiguous sign, read '*plus or minus.*'

$\mp$, ambiguous sign, read '*minus or plus.*'

$\times$, $\cdot$ } signs of multiplication, read '*times*' or '*multiplied by.*'

$\div$, sign of division, read '*divided by*'

$=$, sign of equality, read '*is equal to*' or '*equals.*'

$\equiv$, sign of identity, read '*is identical with.*'

$>$, sign of inequality, read '*is greater than.*'

$<$, sign of inequality, read '*is less than.*'

$\neq$, read '*is not equal to.*'

$\not\equiv$, read '*is not identical with.*'

$\not>$, read '*is not greater than.*'

$\not<$, read '*is not less than.*'

$:$, sign of ratio, read '*is to.*'

$::$, sometimes used between the ratios of a proportion, read '*equals*' or '*as.*'

$\therefore$, sign of deduction, read '*therefore*' or '*hence.*'

$\because$, sign of deduction, read '*since.*'

$\cdots$, sign of continuation, read '*and so on*' or, '*and so on to.*'

(), parentheses }
[], brackets }
{}, braces } signs of aggregation.
—, vinculum }
|, vertical bar }

$\sqrt{\ }$, root, or radical, sign, read '*square root of.*'

$\sqrt[3]{\ }$, $\sqrt[4]{\ }$, etc., read '*cube root of,*' '*fourth root of,*' etc., respectively.

$\sqrt{-1}$, or i, symbol for the imaginary unit.

$\underline{|n}$, factorial sign, read '*factorial n,*' n being any integer.

$\propto$, sign of variation, read '*varies as.*'

∞, symbol of infinity, read '*infinity.*'

0, symbol of an infinitesimal number and of absolute zero, read '*zero.*'

r_1, read '*r-sub one.*'

r_2, read '*r-sub two.*'

r_3, read '*r-sub three.*'

r', read '*r-prime*'.

r'', read '*r-second*'.

r''', read '*r-third.*'

π, symbol for the ratio of the circumference of a circle to its radius, read '*pī.*'

$f(x)$, $F(x)$, $f'(x)$, symbols of functions of x, read '*function of x,*' '*large F function of x,*' and '*f-prime function of x,*' respectively.

SECOND COURSE IN ALGEBRA

INTRODUCTORY REVIEW

1. In this chapter, as well as in all chapters that are entirely or partly review, the student should refer to the Glossary for definitions of terms unfamiliar to him, noting especially terms printed in **black-faced type.**

The algebraic signs and symbols used in this book are explained on page 8.

NOTATION AND DEFINITIONS

EXERCISES

2. Read and tell the meaning of each **algebraic expression**:

1. $r + s.$	6. $xy.$	11. $\sqrt{x}.$	16. $pq + rs.$
2. $a - n.$	7. $z \cdot v.$	12. $\sqrt{2\,rs}.$	17. $7\,x^2 - 3\,y^3.$
3. $2 \times 3.$	8. $4\,x.$	13. $3\,a^2bc^3.$	18. $a^2 - 2\,ab + b^2.$
4. $z \div t.$	9. $5\,y^2.$	14. $(l + t)^2.$	19. $(a + b)(r - s).$
5. $\frac{x - y}{a + b}.$	10. $\frac{s^2}{t^3}.$	15. $\frac{x}{y} - \frac{3}{z}.$	20. $r^4 + 2\,t^2 - 3\,rt^3.$
			21. $x^3 - 3\,x^2 + 3\,x - 1.$

22. How many **terms** has each of the above expressions? Point out the **monomials**; **binomials**; **trinomials**; **polynomials.**

23. Name the **numerical coefficient** in each term in exercises 1–21.

Name the **coefficient** of x in:

24. $3\,x.$ **25.** $lx.$ **26.** $2\,a^3x.$ **27.** $a^2bx.$ **28.** $5\,rs^2t^3x.$

Which coefficient of x is numerical? Which coefficients of x are **literal? mixed?**

Give the numerical coefficient in:

29. x. **30.** $4\,ab$. **31.** $7\,x^2y^3$. **32.** $\frac{1}{2}\,m^2$. **33.** $5\,n^2tx$.

Name the **exponent** of y in:

34. y^2. **35.** $4\,y$. **36.** $2\,xy^3$. **37.** x^2y^2. **38.** $7\,y^5$.

State the difference in meaning between:

39. $2\,x$ and x^2. **40.** $3\,x$ and x^3. **41.** $4\,x$ and x^4. **42.** $5\,x$ and x^5.

43. Write two **similar** monomials; three **dissimilar** monomials.

In $xy + x^2y^2 + ml - 3\,xy - 2\,mx + 4\,x^2y^2 - zy^3$, which terms are **like?** which are **unlike?**

44. What is the value of 1^2? of 1^3? of 1^4? of $\sqrt{1}$? of $\sqrt[3]{1}$? of $\sqrt[4]{1}$? How do these **powers** and **roots** of 1 compare?

45. What is the value of 0×2? of 0×0? of 0^2? of $\sqrt{0}$?

Represent algebraically the:

46. Sum of m and n; sum of the **square** of a and the **cube** of b.

47. Difference of r and t; difference of two times r and three times t.

48. Product of x and y in three different ways; product of the sum and the difference of x and y, using **parentheses**.

49. Quotient of u divided by v in two ways; quotient of $u + v$ divided by $u - v$.

50. Find the cost of 6 apples at y cents each.

51. Grace is a years old. How old will she be in l years?

52. George has m chestnuts and John has n chestnuts. How many more chestnuts has George than John?

53. How long is the side of a square whose perimeter is t feet?

54. If y pounds of tea cost b cents, find the cost of x pounds.

55. Express in brief form $a + a + a + \cdots$ to 8 terms; to n terms.

Order of Operations

3. It is agreed among mathematicians that:

When only + and − occur in any expression, or only × and ÷, the operations are to be performed in order from left to right.

Unless otherwise indicated, as by the use of parentheses:

When ×, ÷, or both, occur in connection with +, −, or both, the indicated multiplications and divisions are to be performed first.

EXERCISES

4. Find the value of:

1. $5 - 3 + 4 + 2 - 3 + 6.$
2. $6 \div 2 \times 7 \div 3 \times 4 \div 2.$
3. $8 - 3 - 5 + 7 + 9 - 4.$
4. $18 \div 6 \times 8 \times 2 \div 12 \times 5.$
5. $8 + 2 \times 4 - 10 + 7 - 9.$
6. $9 - 3 + 8 \div 2 \times 5 - 18.$
7. $6 \times 8 - 4.$
8. $6 \times (8 - 4).$
9. $12 \div 4 + 7 \times 5.$
10. $9 - 3 \times 2 + 8 \div 4.$
11. $(12 - 5) \times 6 \div 3 + 11.$
12. $(12 - 5) \times 6 \div (3 + 11).$

Numerical Substitution

EXERCISES

5. When $a = 3$, $b = 4$, $c = 5$, $n = 2$, find the value of:

1. $8\,b.$
2. $3\,ac.$
3. $5\,cn.$
4. $\dfrac{6\,ac}{an}.$
5. $2\,ab^2.$
6. $(\frac{1}{6}\,an)^3.$
7. $\frac{3}{4}\,bc^3.$
8. $\dfrac{5\,ab^2}{3\,cn}.$
9. $\sqrt{2\,bn}.$
10. $(bc)^2.$
11. $b^2c^2.$
12. $\dfrac{a^nb^3}{4\,a^3n}.$
13. $a\sqrt{bc^2}.$
14. $\sqrt[3]{4\,b^ac^3n}.$
15. $c^2 + n^{b-1}.$
16. $\dfrac{a^2 + b^2}{c^n}.$

When $x = 6$, $y = 3$, $z = 0$, $r = 2$, $s = \frac{1}{2}$, evaluate:

17. $rx + yz + rs - xz.$
18. $sx^2 - r^2s + xyz + xy^2.$
19. $12\,z^3 + r^3y^2 + 5\,xy \div 3\,s.$
20. $\dfrac{4(r+s)^2}{2\,x} \div \dfrac{y+r}{x} \times 4\,s.$
21. $\frac{1}{3}\,xr^2 - \frac{1}{2}\,y^4z + \frac{4}{5}\,s^2.$
22. $4^rsy^2 \div \frac{1}{8}\,x^2r^2 - \frac{11}{12}\,x^3y^2z.$
23. $5\,xy^4 - y^5\sqrt{r^2s^2} + \frac{5}{6}\,xsz.$
24. $x + \left(\dfrac{r+x}{r}\right)\left(\dfrac{z+2\,y^2}{x^r}\right) + s^{y-r}.$

25. The area (A) of any rectangle is equal to the product of the base (b) and the altitude, or height (h).

Write the **formula** for the area of a rectangle in terms of its base and altitude.

Find A when $b=6$ and $h=4$; when $b=12$ and $h=4\frac{1}{2}$.

NOTE. — Since the algebraic form is concerned only with the *number* of units in A, b, and h, in this and similar exercises the principles stated refer only to *numerical* measures.

26. The area of a triangle is equal to one half the product of its base and altitude. Write the formula.

Find A when $b = 15$ and $h = 10$; when $b = 20$ and $h = 7$.

27. The area of a circle is equal to π times the square of its radius (r). ($\pi = 3.1416$, approximately.)

Write the formula and find A when $r = 6$; when $r = .25$.

28. The volume (V) of a rectangular solid is equal to the product of its length (l), breadth (b), and thickness (t).

Write the formula and find V when $l = 7\frac{1}{2}$, $b = 4$, $t = 1\frac{4}{5}$; when $l = 4.5$, $b = 2.4$, $t = .7$.

29. The hypotenuse (c) of a right triangle is equal to the square root of the sum of the squares of the perpendicular sides (a and b). Write the formula.

Find c when $a = 3$ and $b = 4$; when $a = 6$ and $b = 8$.

30. Interest (i) is equal to the principal (p) multiplied by the rate (r) multiplied by the time (t) in years.

Write the formula and find i when $p = 650$, $r = .06$, and $t = 2\frac{1}{3}$.

31. The material removed from the bed of a river in cutting a channel through a bar consisted of s cubic yards of sand and r cubic yards of rock. The cost of removal, per cubic yard, was c cents for sand and d dollars for rock. Express the total cost (T) by a formula.

Find (T), if $s = 300{,}000$, $r = 2000$, $c = 12$, and $d = 4\frac{1}{2}$.

POSITIVE AND NEGATIVE NUMBERS

6. Including zero, the scale of **algebraic numbers** is written:

$$\cdots, -5, -4, -3, -2, -1, 0, +1, +2, +3, +4, +5, \cdots$$

There are as many **negative** numbers *below* zero as **positive** numbers *above* it, zero being neither positive nor negative.

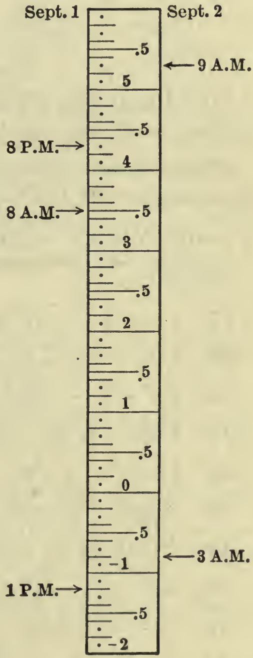
Positive and negative numbers may be added or subtracted by counting forward or backward, respectively, along this scale.

EXERCISES

7. Find the **algebraic sum**, and give its **absolute value**:

1. 5	**2.** 5	**3.** −5	**4.** −5	**5.** −4	**6.** 10
8	−8	8	−8	2	−3

7. −2	**8.** −6	**9.** 20	**10.** −13
−9	16	−7	−13



11–30. In exercises 1–10, subtract the lower number from the upper one; the upper number from the lower one.

The tide gauge shown in the margin is graduated to feet and tenths of a foot. It was nailed to a dock with its zero set at an average stage of low water. By means of it the height of the water in a river was found to vary as follows: Sept. 1, 8 A.M., 3.5 ft.; 1 P.M., − 1.2 ft.; 8 P.M., 4.3 ft.; Sept. 2, 3 A.M., − .8 ft.; 9 A.M., 5.3 ft.

31. How far did the water fall from 8 A.M. to 1 P.M., Sept. 1?

32. How far did the water rise from 1 P.M. to 8 P.M.?

33. How far did the water fall from 8 P.M. to 3 A.M. next day?

34. How far did the water rise from 3 A.M. to 9 A.M.?

ADDITION

8. Law of order, or the **commutative law**, for addition.

Since $3+5=5+3$ and in general $a+b=b+a$,

Numbers may be added in any order.

9. Law of grouping, or the **associative law**, for addition.

Since $4+3+7=(4+3)+7=4+(3+7)=(4+7)+3$ and in general, $a+b+c=(a+b)+c=a+(b+c)=(a+c)+b$,

The sum of three or more numbers is the same in whatever manner the numbers are grouped.

Addition of Monomials

EXERCISES

10. Add:

1.	2.	3.	4.	5.
$7x$	$3r$	$-5an^2$	$-bt^3$	$2x$
$2x$	$-2r$	an^2	$-4bt^3$	$-3y$
$9x$	r	$-4an^2$	$-5bt^3$	$2x-3y$

To add similar monomials, *find the algebraic sum of the numerical coefficients and annex the common literal part.*

When the monomials to be added are dissimilar, they cannot be united into a single term, but their sum may be indicated as in exercise 5.

6–30. Add the monomials in exercises 6–30 on page 16.

Simplify:

31. $5x+3x-6x+x.$

32. $9b-4b+7b+5b.$

33. $\frac{1}{2}a+3a-2\frac{1}{2}a+a.$

34. $3ac-5ac+8ac.$

35. $.4a^2-1.5a^2+8a^2.$

36. $5\sqrt{m}+9\sqrt{m}-\frac{1}{2}\sqrt{m}.$

37. $8a^3b+6a^3b-11a^3b-2a^3b+9a^3b.$

38. $1\frac{4}{5}x^2y^2-\frac{1}{2}x^2y^2-1\frac{7}{10}x^2y^2+3\frac{1}{2}x^2y^2+x^2y^2.$

39. $5(xy)^2-3(xy)^2-15(xy)^2+4(xy)^2+13(xy)^2.$

40. $4\sqrt{abc}+21\sqrt{abc}-8\sqrt{abc}+3\sqrt{abc}-6\sqrt{abc}.$

41. $2(x-1)-13(x-1)+5(x-1)+10(x-1)+6(x-1).$

42. $(a-x)+15(a-x)+7(a-x)-3(a-x)-2(a-x).$

43. $3x(x^2-2x+3)-x(x^2-2x+3)+2x(x^2-2x+3).$

44. $5\sqrt{b-c}+19\sqrt{b-c}-11\sqrt{b-c}+24\sqrt{b-c}-17\sqrt{b-c}.$

Addition of Polynomials

EXERCISES

11. Add:

1. $$\begin{array}{r} 3x-2y \\ -x+5y \\ 2x-8y \\ \hline 4x-5y \end{array}$$

2. $$\begin{array}{r} 2r-3s+5t \\ 3r-4s-t \\ -3r+2s \\ \hline 2r-5s+4t \end{array}$$

3. $$\begin{array}{rrr} a^2+ & 2ab+ & b^2 \\ & -3ab- & b^2 \\ 2a^2+ & ab+ & b^2 \\ \hline 3a^2 & & +b^2 \end{array}$$

RULE. — *Arrange similar terms to stand in the same column. Find the algebraic sum of each column, and write the results in succession with their proper signs.*

4. $$\begin{array}{r} 3b+4c \\ -b+2c \\ 5b-8c \\ \hline \end{array}$$

5. $$\begin{array}{rrr} 2l+ & m- & n \\ -l+ & 4m+ & 5n \\ & -5m- & 2n \\ \hline \end{array}$$

6. $$\begin{array}{rrr} 3c^2- & 4cd+ & d^2 \\ 4c^2+ & 5cd & \\ -c^2+ & 2cd- & 6d^2 \\ \hline \end{array}$$

7. $4a+3b,\ -5b.$
8. $2x-4y,\ -3x.$
9. $3d+4c,\ 2c-5d.$
10. $5x^2+y^2,\ y^2-3x^2.$
11. $2m+n,\ 3n-6m.$
12. $2a^2b+c^2,\ 2c^2-a^2b.$
13. $4r-5t,\ 2r-s+3t,\ 2s-3r,$ and $s+2t.$
14. $a-\frac{1}{2}b+c,\ \frac{3}{2}a+\frac{1}{3}b,\ \frac{1}{2}a-\frac{1}{3}c,$ and $\frac{2}{3}b-\frac{2}{3}c.$
15. $\sqrt{a}+\sqrt{ab},\ 2\sqrt{a}-\sqrt{b}+2\sqrt{ab},$ and $2\sqrt{b}-3\sqrt{ab}-\sqrt{a}.$
16. $x+3(a+1)-y,\ -(a+1)-2x+4y,$ and $3x-4(a+1).$

Simplify the following polynomials:

17. $4a-b+2c+3b-d-3a+3c+2d-4b+5a-4c-3d.$
18. $3x^a-2y^b+1+3y^b-4x^a+y^b-6+2x^a+4-2y^b+x^a.$
19. $(a+c)x-(b-d)y-2(a+c)x+3(b-d)y+4(a+c)x-(b-d)y.$
20. $2ay-3ac-4ay+4ac-6ay+5ac+11ay-4ac-ay.$
21. $2c-7d+6n+11m-3c-5n+8d-3m+10c+7n-2d-8c+4d-3n-8m-6n+m-3d+2m.$
22. $5am-3a^2m^2+4-4am+a^2m^2-2+5+a^2m^2-6+3am+2a^2m^2-3am-3+8am+2-a^2m^2.$
23. $6\sqrt{x}-5\sqrt{xy}+3\sqrt{y}-4\sqrt{x}+6\sqrt{xy}-\sqrt{x}-\sqrt{y}+3\sqrt{y}-2\sqrt{xy}+\sqrt{x}+2\sqrt{xy}-3\sqrt{y}+6\sqrt{x}+4\sqrt{xy}+\sqrt{y}.$

SUBTRACTION

Subtraction of Monomials

EXERCISES

12. Subtract the lower monomial from the upper one:

1.	2.	3.	4.	5.
$12a$	$8b$	$3d^2$	$-7c^3x$	$3(a-b)$
$5a$	$-4b$	$2d^3$	$-2c^3x$	$-2(a-b)$
$7a$	$12b$	$3d^2-2d^3$	$-5c^3x$	$5(a-b)$

RULE. — *Consider the sign of the subtrahend to be changed, and add the result to the minuend.*

6.	7.	8.	9.	10.
$7x$	$7x$	$7x$	$7x$	$7x$
$5x$	$6x$	$7x$	$8x$	$9x$

11.	12.	13.	14.	15.
4	a	x	a	$-m$
-1	1	-2	b	$-2n$

16.	17.	18.	19.	20.
a^2b	mnx	$-2ab^3$	$-\sqrt{r}$	$-5(x+y)$
$-a^2b$	$2mnx$	$-4ab^3$	$3\sqrt{r}$	$9(x+y)$

21.	22.	23.	24.	25.
$5xy$	$-8b$	$5n^2r$	0	0
$2sn$	$2cd$	$-4w$	$2(a+b)$	$(x-y)$

26.	27.	28.	29.	30.
$5a^n$	$-15b^2c^2$	$-7x^2y^3$	$-13\sqrt{x}$	$-3(a+b)$
$-2a^n$	$9b^2c^2$	$-14x^2y^3$	$-5\sqrt{x}$	$-10(a+b)$

31–55. In each of the exercises 6–30, subtract the upper monomial from the lower one.

Subtraction of Polynomials

EXERCISES

13. Subtract the lower expression from the upper one:

1. $\begin{array}{r} a+b+c \\ \underline{a-b+c} \\ 2b \end{array}$ 2. $\begin{array}{r} 3m-2n-\ \ p^2 \\ \underline{-m+\ \ n-6p^2} \\ 4m-3n+5p^2 \end{array}$ 3. $\begin{array}{r} x^2+2xy-\ \ y^2 \\ \underline{-x^2-3xy+\ \ y^2} \\ 2x^2+5xy-2y^2 \end{array}$

RULE. — *Arrange similar terms to stand in the same column. Consider the sign of each term of the subtrahend to be changed, and add the result to the minuend.*

4. $\begin{array}{r} a+b \\ \underline{a-b} \end{array}$ 5. $\begin{array}{r} x-y \\ \underline{x+y} \end{array}$ 6. $\begin{array}{r} -a+m \\ \underline{a-m} \end{array}$ 7. $\begin{array}{r} 2r-\ \ s \\ \underline{r-2s} \end{array}$

8. $\begin{array}{l} 1 \\ \underline{1-x-x^2} \end{array}$ 9. $\begin{array}{l} x \\ \underline{2x-x^2+4} \end{array}$ 10. $\begin{array}{r} 5 \\ \underline{x^2-3x+5} \end{array}$ 11. $\begin{array}{r} -x+1 \\ \underline{a+x-2} \end{array}$

12–19. In exercises 4–11, subtract the upper expression from the lower one.

20. From a^4+1 subtract $1-a+a^2-a^3+a^4$.

21. What number subtracted from $a-x$ will give $a+x$?

22. Take $4x^m+2x^my^n+5y^n$ from $7x^m+2x^my^n+9y^n$.

23. To what must r^2-4s^2 be added to produce $3s^2-r^2$?

24. From $5x-2y$ subtract the sum of $2x-y$ and $x-2y$.

25. Take $6m^s+11m^sn^t+5n^t$ from $10m^s+11m^sn^t+8n^t$.

26. From the sum of $1+x$ and $1-x^2$ subtract $1-x+x^2-x^3$.

27. What number added to $a^2-ab-ac+b^2-c^2$ will give 0?

28. From the sum of $a-b-c$ and $a+b+c$ subtract the sum of $a-b+c$ and $a+b-c$.

29. From the sum of $3x^2-2x+1$ and $2x-5$ subtract the sum of $x-x^2+1$ and $2x^2-4x+3$.

If $r=c^2-d^2$, $s=c^2+d^2$, $t=c^2+2cd+d^2$, and $u=2cd$,

30. $r-s+t+u=?$ **32.** $r-s+t-u=?$

31. $r+s-t+u=?$ **33.** $s-r-u+t=?$

PARENTHESES

Removal of Parentheses

14. When numbers are included by any of the **signs of aggregation**, they are commonly said to be *in parenthesis, in a parenthesis*, or *in parentheses*.

15. The sign + before a parenthesis indicates that the terms in parenthesis are to be added and the sign −, that they are to be subtracted. Hence,

PRINCIPLES. — 1. *A parenthesis preceded by a plus sign may be removed from an expression without changing the signs of the terms in parenthesis.*

2. *A parenthesis preceded by a minus sign may be removed from an expression, if the signs of all the terms in parenthesis are changed.*

EXERCISES

16. Simplify each of the following:

1. $b+(-a)$.	**8.** $x+v-(y-z)$.
2. $x+(y-z)$.	**9.** $a+c-(a+d)$.
3. $l-(r-s)$.	**10.** $a-b-(-c+a)$.
4. $m-(m-n)$.	**11.** $l-(t+v)+(u+v)$.
5. $a+(-b+c)$.	**12.** $.5x-a+(1.5x+a)$.
6. $4c+(d-2c)$.	**13.** $3x^2+xy-(y^2+2xy+x^2)$.
7. $a-(-b+2a)$.	**14.** $a+b-(2a+2b)+(4b-a)$.

When an expression contains parentheses within parentheses, they may be removed *in succession*, beginning with either the outermost or the innermost, preferably the latter.

15. Simplify $6x-[3a-\{4b+(8b-2a)-3b\}+4x]$.

SOLUTION.	$6x-[3a-\{4b+(8b-2a)-3b\}+4x]$
Prin. 1,	$=6x-[3a-\{4b+8b-2a-3b\}+4x]$
Prin. 2,	$=6x-[3a-4b-8b+2a+3b+4x]$
Prin. 2,	$=6x-3a+4b+8b-2a-3b-4x$
Uniting terms,	$=2x-5a+9b$.

Simplify each of the following:

16. $a+2b+(14a-5b)-\{6a+6b-(a+4b)\}.$

17. $12a-\{4-3b-(6b+3c)+b-8-(5a-2b-6)\}.$

18. $25-[10-\overline{11-7}-(16-14)+\overline{8+6-3}].$

19. $x^3-[x^2-(1-x)]-\{1+x^2-(1-x)+x^3\}.$

20. $1-x-\{1-[x-1+(x-1)-(1-x)-x]+1-x\}.$

21. $-\{3ax-[5xy-3z]+z-(4xy+[6z+7ax]+3z)\}.$

22. $1-\{a-[-2a+a^2-(a^2+a^3)-4a^2]+[1-(3a+4a^2)]\}.$

Grouping Terms in Parentheses

17. It follows from the principles in § 15 that:

Principles. — 1. *Any number of terms of an expression may be inclosed in a parenthesis preceded by a plus sign without changing the signs of the terms to be inclosed.*

2. *Any number of terms of an expression may be inclosed in a parenthesis preceded by a minus sign, provided the signs of the terms to be inclosed are changed.*

In grouping terms, it is customary to make the first term of each group positive by choosing the proper sign, + or −, to precede the group.

EXERCISES

18. Group as binomials without changing order of terms:

1. $a+b+c-d.$
2. $a-b-c-d.$
3. $a+b-c+d.$
4. $x^2-y^2-xy+y^2-2x^3-2y^3.$
5. $1-x+x^2-x^3-x^4+x^5-x^6+x^7.$
6. $1-2x-4x^2+8x^3-16x^4-32x^5.$

Group the last three terms as a subtrahend:

7. $x^2-y^2+2yz-z^2.$
8. $c^2-b^2-2bd-d^2.$
9. $y^4+v^4-x^4+2x^2z-z^2.$
10. $c^2+2cd+d^2-a^4-a^3+a^2.$

Group the terms of **like degree** beginning with the highest:

11. $a^2+a-b+b^2.$
12. $2-x^2+2xy-y^2.$
13. $r^3+s^3-3rs+s^2.$
14. $x^2-2xy+y^2-2x+2y.$
15. $c+c^3-c^2d-d^2+cd+d.$
16. $a^4+4a^3b-b^2+a-ab+b.$

Collecting Coefficients

EXERCISES

19. Add:

1. $\begin{array}{r} ry \\ sy \\ \hline (r+s)y \end{array}$ **2.** $\begin{array}{r} sw \\ -tw \\ \hline (s-t)w \end{array}$ **3.** $\begin{array}{r} -ax \\ -bx \\ \hline -(a+b)x \end{array}$ **4.** $\begin{array}{r} (b-2c)v \\ (b+c)v \\ \hline (2b-c)v \end{array}$

5. $\begin{array}{r} -4m \\ -bm \\ \hline \end{array}$ **6.** $\begin{array}{r} -ay \\ 2by \\ \hline \end{array}$ **7.** $\begin{array}{r} 2pq \\ -5q \\ \hline \end{array}$ **8.** $\begin{array}{r} (2c-d)z \\ (d-3e)z \\ \hline \end{array}$

9–16. In exercises 5–8, subtract the lower expression from the upper one; the upper expression from the lower one.

Simplify:

17. $(a+c)x+(a-c)x$.

18. $(a+c)x-(a-c)x$.

19. $(b-d)y-(b+d)y$.

20. $(a+2)z+(3-a)z$.

21. $(4+c^2)v-(2c^2+2)v$.

22. $(a-b)s^2+(2b-a)s^2$.

23. $(a^2+b^2)x^2-(a^2-b^2)x^2$.

24. $(2t^2+1)y^2+(4-3t^2)y^2$.

25. Collect the coefficients of x and of y in $ax-by-bx-ay$.

Solution. — The total coefficient of x is $(a-b)$.

The total coefficient of y is $(-a-b)$, or $-(a+b)$.

$\therefore ax-by-bx-ay=(a-b)x-(a+b)y$.

Collect the coefficients of x and of y in:

26. $ax-bx+cy+dy$.

27. $mx-nx-ry+sy$.

28. $ax-3x+dy-4y$.

29. $nx-ny-2x+5y$.

30. $2ax-2ay-x-y$.

31. $2cx-ay+by-3dx$.

32. $3ax+2ay-by-2bx$.

33. $Ax+2By-A'x-B'y$.

34. $3rx-5my+25x-ny$.

35. $7px-4qy-12x+10y$.

36. $a^2x-b^2y-2ax+2by+x+y$.

37. $(a^2-1)x-(a^2+1)y-2ax+4ay+2x-3y$.

38. $b^3x-n^3y-3b^2x+3n^2y-3bx+3ny-x-y$.

39. $(a^2-4a+2)x+(a^2-6a+1)y-(a^2-3a)x+8y$.

40. $(5c^2-2d)x-(3c^2+4d)y-(4c^2+1)x+(2c^2+4)y$.

MULTIPLICATION

20. **Law of order,** or the **commutative law,** for multiplication.

Since $2 \times 3 = 3 \times 2$, and in general $ab = ba$,

The factors of a product may be taken in any order.

21. **Law of grouping,** or the **associative law,** for multiplication.

Since $2 \times 3 \times 5 = (2 \times 3) \times 5 = 2 \times (3 \times 5) = (2 \times 5) \times 3 = (3 \times 5) \times 2$, and in general $abc = (ab)c = a(bc) = (ac)b = (bc)a$,

The factors of a product may be grouped in any manner.

Multiplication of Monomials

EXERCISES

22. Multiply:

1.	2.	3.	4.	5.
a	$-x$	$-2r^2$	$5ab^2c^3$	$-4x^m$
b	$2y$	$-3r^3$	$-2a^3b^2c$	$-3x^n$
ab	$-2xy$	$6r^5$	$-10a^4b^4c^4$	$12x^{m+n}$

In finding the product of two monomials, apply in succession the following *laws for multiplication:*

Law of signs. — *The sign of the product is + when the multiplicand and multiplier have* ***like signs,*** *and − when they have* ***unlike signs.***

Law of coefficients. — *The coefficient of the product is equal to the product of the coefficients of the multiplicand and multiplier.*

Law of exponents. — *The exponent of a number in the product is equal to the sum of its exponents in the multiplicand and multiplier.*

6.	7.	8.	9.	10.
$4a^2$	$-6m^4n^3$	$-p^2q$	$a^2b^3x^3y^n$	$-2a^2m^3n^4$
-1	$-3n^2m^3$	ap^2q^3	$ab^nx^2y^2$	$8b^4n^4m^7$

11.	12.	13.	14.	15.
$-x^2y^3z$	$-ab^2c^3$	$-4a^rb^s$	$a^{n-1}b^{n-2}c^3$	$m^an^cb^2y^a$
$5xy^5z^4$	$d^mb^{2n}c$	$-3a^{r-2}b^s$	$a^{n+1}b^2c^{n-1}$	$m^bn^db^xy^{b-a}$

23. **When there are several monomials**, by the law of signs,

$$-a \times -b = +ab;$$
$$-a \times -b \times -c = +ab \times -c = -abc;$$
$$-a \times -b \times -c \times -d = -abc \times -d = +abcd; \text{ etc.} \quad \text{Hence,}$$

The product of an even number of negative factors is positive; of an odd number of negative factors, negative.

Positive factors do not affect the sign of the product.

EXERCISES

24. Find the products indicated:

1. $(-1)(-1)(-1)$.
2. $(-2)(-ab)(-3a^2)$.
3. $(-a^2x)(4bx)(-5a^2)$.
4. $(-2xy)(-3xy)(5x^2)(-y^2)$.
5. $(-4bc)b(-3c^2)c(-b)(-c)$.
6. $(-2^3)(-2^4)(5 \cdot 2^2)(-5^2 \cdot 2)$.

Multiplication of Polynomials by Monomials

25. The **distributive law** for multiplication.

In general, $a(x+y+z) = ax + ay + az$. That is,

The product of a polynomial by a monomial is equal to the algebraic sum of the partial products obtained by multiplying each term of the polynomial by the monomial.

EXERCISES

26. Multiply as indicated:

1. $2a(3x+2y)$.
2. $-l(5w-uv)$.
3. $-3b(4c+3e)$.
4. $a^2bc(3a^4-4a^3b)$.
5. $2xy(5x^2-10xy)$.
6. $5m^3(6m^3-2m^2n)$.
7. $a^{2n}(3x^3-10a^ny^2)$.
8. $ax^2(x^n-x^{n-1}+x^{m-2})$.
9. $3tu^2(u^3+4t^3-2t^3u^3)$.
10. $-xyz(-xy+yz+2xz)$.
11. $-3yz(y^3-3y^2z^2-3yz^3+z^4-y^4+3y^3z)$.
12. $abc(a^2b^2-2a^2c^2-2b^2c^2-a^4-4b^4-c^4-5abc)$.
13. $-bc(b^4+c^4-b^3-c^3+b^2c^2-4b^2c+8bc^2-2bc)$.
14. $m^an^3(m^4-5m^3n^b-16m^2n^{2b}+24mn^{3b}-n^{4b})$.
15. $x^{n-3}y^{m+4}(x^3y^{m-3}-5x^{4-n}y^{m-2}+10x^{5-n}y^{m-1}-5x^{4-2n}y^{2-m})$.

Multiplication of Polynomials

EXERCISES

27. **1.** Multiply $a^2 - ay + y^2$ by $a + y$.

$$
\begin{array}{llcc}
\text{PROCESS} & & & \text{TEST (When } a = 2 \\
a^2 - ay + y^2 & & = & 7 \qquad \text{and } y = 3) \\
\underline{a + \ y \qquad\qquad} & & = & \underline{\ 5\ } \\
a^3 - a^2y + ay^2 & & & \\
\underline{\qquad a^2y - ay^2 + y^3} & & & \\
a^3 \qquad\qquad\quad + y^3 & & = & 35
\end{array}
$$

RULE. — *Multiply the multiplicand by each term of the multiplier and find the algebraic sum of the partial products.*

TEST. — To test the result, assign to each letter any value, and observe whether for these values,

Product obtained = *multiplier* × *multiplicand.*

It is usually most convenient to substitute 1 for each letter, but since any power of 1 is 1, such a value does not test the exponents.

Multiply as indicated, and test:

2. $(3a + 4)(a + 2)$.
3. $(2x + 1)(3 + x)$.
4. $(2a + 4)(4a - 3)$.
5. $(6b + 1)(2b - 4)$.
6. $(5c + 2d)(2c + d)$.
7. $(2x - 3a)(3x - 4a)$.
8. $(ab - 15)(ab + 10)$.
9. $(3a + cd)(4a + cd)$.
10. $(x^2 + x + 1)(x - 1)$.
11. $(y^2 + by - b^2)(y - b)$.
12. $(a + b + c)(a + b - c)$.
13. $(x - y + z)(x + y - z)$.
14. $(x^{2n} + 2x^ny^m + y^{2m})(x^n - y^m)$.
15. $(5x - 5x^2 + 10)(12 - 30x + 2x^2)$.
16. $(4x - 3x^2 + 2x^3)(3x - 10x^2 + 10)$.
17. $(2a^2 - 3b^2 - ab)(3a^2 - 4ab - 5b^2)$.
18. $(a^{2n} - 2a^nb^n + b^{2n})(a^{2n} + 2a^nb^n + b^{2n})$.
19. $(a^2 + b^2 + c^2 - ab - ac - bc)(a + b + c)$.
20. $(x^{a+1}y^{c-1} + x^{c-1}y^{a+1} + 1)(x^{c-1}y^{a+1} - x^{a+1}y^{c-1} + 1)$.
21. $(\frac{1}{2}z^{2a+1} - \frac{1}{2}z^{2a} + \frac{1}{2}z^{2a-1})(2z^{2a-1} + 2z^{2a-2} + 2z^{2a-3})$.

28. When polynomials are **arranged** according to the **ascending** or the **descending powers** of some letter, processes may often be abridged by using the *detached coefficients.*

EXERCISES

29. **1.** Expand $(2x^4-3x^3+3x+1)(3x+2)$.

FULL PROCESS

$$\begin{array}{l}
2x^4-3x^3+3x+1\\
3x+2\\
\hline
6x^5-9x^4\qquad\quad+9x^2+3x\\
\qquad 4x^4-6x^3\qquad\quad+6x+2\\
\hline
6x^5-5x^4-6x^3+9x^2+9x+2
\end{array}$$

DETACHED COEFFICIENTS

$$\begin{array}{rrrrrr}
2 & -3 & +0 & +3 & +1 & \\
3 & +2 & & & & \\
\hline
6 & -9 & +0 & +9 & +3 & \\
 & 4 & -6 & +0 & +6 & +2\\
\hline
\multicolumn{6}{l}{6x^5-5x^4-6x^3+9x^2+9x+2}
\end{array}$$

Since the second power of x is wanting in the first factor, the term, if it were supplied, would be $0x^2$. Therefore, the detached coefficient of the term is 0. The detached coefficients of missing terms should be supplied to prevent confusion in placing the coefficients in the partial products and to avoid errors in writing the letters in the result.

Arrange properly and expand, using detached coefficients:

2. $(x+x^3+1+x^2)(x-1)$.

3. $(x^3+10-7x-4x^2)(x-2)$.

4. $(14-9x-6x^2+x^3)(x+1)$.

5. $(a^3+4a^2-11a-30)(a-1)$.

6. $(4a^2-8a+a^4-3)(2+a)$.

7. $(2m-3+2m^3-4m^2)(2m-3)$.

8. $(x+x^2-5)(x^2-3-2x)$.

9. $(b^2+5b-4)(-4+2b^2-3b)$.

10. $(y^3-5y+2y^4+8)(2y^2+y+1)$.

11. $(4n^3+6-2n^4+16n-8n^2+n^5)(n+2)$.

12. $(7+5x^2-4x^3+3x^4-2x^5+x^6)(x^2+2x+1)$.

13. $(1+x+4x^2+10x^3+46x^5+22x^4)(2x^2+1-3x)$.

Special Cases in Multiplication

30. Show the truth of each of the following formulas by actual multiplication, and state the corresponding principle in words:

Formula 1. $$(a+b)^2 = a^2 + 2\,ab + b^2.$$

APPLICATIONS. $(x+3)^2 = x^2 + 6\,x + 9.$

Also, $14^2 = (10+4)^2 = 10^2 + 2 \times 10 \times 4 + 4^2 = 196.$

Formula 2. $$(a-b)^2 = a^2 - 2\,ab + b^2.$$

APPLICATIONS. $(2-y)^2 = 4 - 4\,y + y^2.$

Also, $19^2 = (20-1)^2 = 20^2 - 2 \times 20 \times 1 + 1^2 = 361.$

Formula 3. $$(a+b)(a-b) = a^2 - b^2.$$

APPLICATIONS. $(x+5)(x-5) = x^2 - 25.$

Also, $32 \times 28 = (30+2)(30-2) = 30^2 - 2^2 = 896.$

Formula 4. $$(x+a)(x+b) = x^2 + (a+b)x + ab.$$

APPLICATIONS. $(x+2)(x+5) = x^2 + 7\,x + 10.$

Also, $(y+1)(y-4) = y^2 - 3\,y - 4.$

Also, $(n-2)(n-3) = n^2 - 5\,n + 6.$

Formula 5. $$(a+b+c)^2 = a^2 + b^2 + c^2 + 2\,ab + 2\,ac + 2\,bc.$$

APPLICATION. $(x-y+3\,z)^2 = x^2 + y^2 + 9\,z^2 - 2\,xy + 6\,xz - 6\,yz.$

EXERCISES

31. Expand by inspection:

1. $(x+y)^2$.
2. $(r-s)^2$.
3. $(a+4)^2$.
4. $(x-3)^2$.
5. $(5+y)^2$.
6. $(a^2+b^2)^2$.
7. $(x^3-y^3)^2$.
8. $(y^r+1)^2$.
9. 21^2.
10. 18^2.
11. 32^2.
12. 43^2.
13. 28^2.
14. 52^2.
15. 27^2.
16. 34^2.
17. $(22)(18)$.
18. $(r+s)(r-s)$.
19. $(t+2)(t+3)$.
20. $(3\,r+4)(3\,r+4)$.
21. $(ab+cd)(ab-cd)$.
22. $(x^{a+1}+y^b)(x^{a+1}-y^b)$.
23. $(\overline{a+b}+c)(\overline{a+b}-c)$.
24. $(a-b+c)(a-b+c)$.

Expand by inspection:

25. $(x^n + y^n)^2$.
26. $(3 - 2y^2)^2$.
27. $(a^n + b^m)^2$.
28. $(1 - 3xy)^2$.
29. $(2x^4 + .5)^2$.
30. $(r^3 - .2s)^2$.
31. $(y^m - y^n)^2$.
32. $(4a + .3b)^2$.
33. $(1 + r^a s^b)^2$.
34. $(.2x^3 - 5)^2$.
35. $(\frac{1}{2}x^2 + a^2)^2$.
36. $(\frac{1}{3}m^2 - \frac{3}{4}n)^2$.
37. $(abc - \frac{1}{2}d)^2$.
38. $(\frac{3}{4}x - \frac{2}{3}y)^2$.
39. $(a + b - c)^2$.
40. $(x + y + z)^2$.
41. $(x + 3)(x - 4)$.
42. $(2x + 1)(2x + 2)$.
43. $(\frac{1}{2}a - 2)(\frac{1}{2}a - 4)$.
44. $(ax - 1)(ax + 3)$.
45. $(xy + 4)(xy - 4)$.
46. $(a^{2n} - b^m)(a^{2n} + b^m)$.
47. $(3r + 2)(2r + 10)$.
48. $(5x - .12)(5x + .12)$.

49. $(b - c - d)^2$.
50. $(3x^2y^2 + 2a^nz^2)^2$.
51. $(\frac{1}{4}rs - \frac{1}{2}t^{a+1})^2$.
52. $(ac + bd - ce)^2$.
53. $(m^{a+2} - m^b n^{c+1})^2$.
54. $(2.5 + 12p^2q^2r^3)^2$.
55. $(2a + 3b + 4)^2$.
56. $(x^{m-n} - y^{m+n})^2$.
57. $(a^2b^2c^2 + d^2e^2)^2$.
58. $(x^2 - 3y^3 - 2z^4)^2$.
59. $(\frac{2}{3}m^4n^3 + \frac{3}{4}p^2q^2)^2$.
60. $(3an + 4ab + 6)^2$.
61. $(4def - \frac{1}{4}abc)^2$.
62. $(5r^2s^3 + 1.5r^{a-1})^2$.
63. $(18 - 2a + 3bc)^2$.
64. $(x + 5b)(x - 5b)$.
65. $(3x + 4)(3x - 5)$.
66. $(2a + b)(3a - 2b)$.
67. $(2a + y)(2a + x)$.
68. $(2y + z)(3y - 2z)$.
69. $(y^2z^2 - 8)(y^2z^2 + 5)$.
70. $(5r^2 + 2s)(2r^2 - 5s)$.
71. $(3x^n + m^n)(3x^n + n^m)$.
72. $(r + \overline{s + t})(r - \overline{s + t})$.
73. $(5cdx + 1)(5cdx - 6)$.
74. $(3bx^3 + 7)(3 + 7bx^3)$.
75. $(ad^2x^3 - 10)(ad^2x^3 - 3)$.
76. $(x^{a+b} + 3y)(x^{a+b} - 2y)$.
77. $(t + u + v)(t - u - v)$.
78. $(a^mb^n + x^y)(a^mb^n - x^y)$.
79. $(x^{a-1} - y)(2x^{a-1} + 3y)$.
80. $(7a^2x^3 + 6z^n)(7a^2x^3 - 6z^n)$.
81. $(a^{m-1} + b^{n-1})(a^{m-1} + b^{n-1})$.
82. $(4az^3 - 3y^2)(2az^3 + y^2)$.
83. $(x^ay^b + x^by^a)(x^ay^b - x^by^a)$.
84. $(\overline{a + b} + \overline{c + d})(\overline{a + b} - \overline{c + d})$.

DIVISION

Division by Monomials

EXERCISES

32. Divide as indicated:

1. $a\overline{)-ab}$ / $-b$ 2. $-5x\overline{)25xy^2}$ / $-5y^2$ 3. $5^n\overline{)5^m}$ / 5^{m-n} 4. $-4c^2d^4\overline{)-c^4d^5}$ / $\frac{1}{4}c^2d$

In finding the quotient of two monomials, apply in succession the following *laws for division:*

Law of signs. — *The sign of the quotient is + when the dividend and divisor have* **like signs**, *and — when they have* **unlike signs.**

Law of coefficients. — *The coefficient of the quotient is equal to the coefficient of the dividend divided by the coefficient of the divisor.*

Law of exponents. — *The exponent of a number in the quotient is equal to its exponent in the dividend minus its exponent in the divisor.*

Since a number divided by itself equals 1, $a^5 \div a^5 = a^{5-5} = a^0 = 1$; that is, a number whose exponent is 0 is equal to 1. (Discussed in § 175.)

5. $2^2\overline{)2^3}$. 10. $-x\overline{)x^5}$. 15. $4s\overline{)-8s^2}$. 20. $8a^4b^5 \div -4a^2b^2$.

6. $2^2\overline{)2^4}$. 11. $z^2\overline{)x^2z^3}$. 16. $-2n^3\overline{)6n^3}$. 21. $-20b^5y^2 \div 5b^2y$.

7. $3^4\overline{)3^4}$. 12. $2\overline{)4m}$. 17. $7l^4\overline{)-14l^8}$. 22. $-6a^4y^4 \div -9a^3y^2$.

8. $5^2\overline{)5^4}$. 13. $-2\overline{)r^2}$. 18. $2\pi r\overline{)4\pi r^2}$. 23. $-4x^mz^r \div 32x^nz^s$.

9. $\dfrac{a^{10}b^4}{a^5b}$. 14. $\dfrac{3x^2y^3z}{-xy^2}$. 19. $\dfrac{4a^4b^3c^5}{20a^2bc^3}$. 24. $\dfrac{2a^2(x-y)^3}{-a(x-y)^2}$.

33. The **distributive law** for division.

Since $(a+b)x = ax + bx$, $(ax+bx) \div x = a + b$; that is,

The quotient of a polynomial by a monomial is equal to the algebraic sum of the partial quotients obtained by dividing each term of the polynomial by the monomial.

EXERCISES

34. Divide:

1. $-xy)\underline{ax^2y - 2\,xy^2}$
   $\quad\quad\; -ax + 2\,y$.

2. $3\,ax^2)\underline{9\,a^2x^3 - 12\,a^3x^5 + 6\,ax^4}$
   $\quad\quad\quad\; 3\,ax - \;\; 4\,a^2x^3 + 2\,x^2$.

3. $6\,a^2b^2 - 9\,ab^3$ by $3\,ab$.

4. $4\,x^2y^2 + 2\,x^2y^3$ by $2\,x^2y^2$.

5. $abc^2 - 2\,a^2b^2c$ by $-\,abc$.

6. $9\,x^2y^2z + .3\,xyz^2$ by $.3\,xy$.

7. $-\,a - b - c - d$ by $-\,1$.

8. $-\,a + a^2b - a^2c$ by $-\,a$.

9. $x^2y - xy^2 + x^2y^3$ by $\frac{1}{2}\,xy$.

10. $c^2d - 3\,cd^2 + 4\,c^3d^3$ by $-\,cd$.

11. $34\,a^nx^6y^2 - 51\,a^{n+2}x^4y^4 - 68\,a^{n+4}x^2y^6$ by $17\,a^2x^2y^2$.

12. $8\,a^7b^{x+4} - 28\,a^6b^{x+3} - 16\,a^5b^{x+2} + \;a^4b^{x+1}$ by $4\,a^4b^3$.

13. $2\,a^2(b-c)^3 - 3\,ab\,(b-c)^2 + 2\,bc\,(b-c)$ by $(b-c)$.

14. $3\,(x-y) - 3\,x\,(x-y)^2 + 4\,x^2\,(x-y)^3$ by $(x-y)$.

15. $r^{8m}s^{6n} - 3\,r^{7m}s^{7n} + 5\,r^{6m}s^{8n} - 6\,r^{5m}s^{9n}$ by $-\,5\,r^{4m}s^{6n}$.

16. $a^{2x+4}b^{x+2} - a^{2x+3}b^{x+4} + a^{2x+2}b^{x+6} - a^{2x+1}b^{x+8}$ by $a^{2x}b^{x+1}$.

Division by Polynomials

EXERCISES

35. 1. Divide $2\,a^4 - 5\,a^3b + 6\,a^2b^2 - 4\,ab^3 + b^4$ by $a^2 - ab + b^2$.

PROCESS

division

$$
\begin{array}{l|l}
2\,a^4 - 5\,a^3b + 6\,a^2b^2 - 4\,ab^3 + b^4 & a^2 - \;\; ab + b^2 \\
\underline{2\,a^4 - 2\,a^3b + 2\,a^2b^2} & \overline{2\,a^2 - 3\,ab + b^2} \\
\quad - 3\,a^3b + 4\,a^2b^2 - 4\,ab^3 & \\
\quad \underline{- 3\,a^3b + 3\,a^2b^2 - 3\,ab^3} & \\
\quad\quad\quad\quad\quad a^2b^2 - \;\; ab^3 + b^4 & \\
\quad\quad\quad\quad\quad \underline{a^2b^2 - \;\; ab^3 + b^4} &
\end{array}
$$

answer

TEST

$-7 \div 7 = -1$

(When $a = 2$ and $b = 3$.)

NOTE. — When $a = 1$ and $b = 1$ the test becomes $0 \div 1 = 0$. In general, $0 \div a = 0$; that is, zero divided by any number equals zero.

Similarly, the result may be tested by substituting any other values for a and b, except such values as give for the result $0 \div 0$, or any number divided by 0, for reasons that will be shown in § 283.

RULE. — *Arrange both dividend and divisor according to the ascending or the descending powers of a common letter.*

Divide the first term of the dividend by the first term of the divisor, and write the result for the first term of the quotient.

Multiply the whole divisor by this term of the quotient, and subtract the product from the dividend. The remainder will be a new dividend.

Divide the new dividend as before, and continue to divide in this way until the first term of the divisor is not contained in the first term of the new dividend.

If there is a remainder after the last division, write it over the divisor in the form of a fraction, and add the fraction to the part of the quotient previously obtained.

Divide, and test each result:

2. $x^3 + x^2y + xy^2 + y^3$ by $x + y$.
3. $6a^2 + 13ab + 6b^2$ by $3a + 2b$.
4. $3m^2 - 4am^3 + a^2m^4$ by $am - 1$.
5. $x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$ by $x - y$.
6. $a^3 + 5a^2x + 5ax^2 + x^3$ by $a^2 + 4ax + x^2$.
7. $a^6 + 5a^5 - a^3 + 4a^4 + 2a - 3a^2 + 3$ by $a - 1$.

8. $c^4 - d^4$ by $c - d$.
9. $x^4 + y^4$ by $x + y$.
10. $r^3 - s^3$ by $r + s$.
11. $a^5 - b^5$ by $a - b$.
12. $x^4 + 81$ by $x - 3$.
13. $a^7 + b^7$ by $a + b$.
14. $x^6 - 64$ by $x + 2$.
15. $m^8 + n^8$ by $m^2 + n^2$.

16. $x^{3n-3} + y^{3n+3}$ by $x^{n-1} + y^{n+1}$.
17. $x^3 + y^3 + z^3 - 3xyz$ by $x + y + z$.
18. $a^3 - b^3 + c^3 + 3abc$ by $a^2 + b^2 + c^2 + ab - ac + bc$.
19. $\frac{9}{16}m^4 + \frac{4}{3}m - \frac{3}{4}m^3 + \frac{16}{9} - \frac{7}{4}m^2$ by $\frac{3}{2}m^2 - m - \frac{8}{3}$.
20. $\frac{1}{2}a^2x^2 - \frac{3}{2}ax^3 + \frac{9}{8}x^4 - \frac{2}{9}a^4$ by $\frac{3}{4}x^2 + \frac{1}{3}a^2 - \frac{1}{2}ax$.
21. $x^n + y^n$ by $x + y$ to five terms of the quotient.
22. $-x^{2r+1}y^{2s} - 2x^{2r+3}y^{2s+1} - x^{2r+5}y^{2s+2}$ by $-x^ry^{s-1} - x^{r+2}y^s$.

Divide, using detached coefficients:

23. $x^3 + 4x^2 + 7x + 6$ by $x - 2$.

PROCESS

$$
\begin{array}{llll}
1 + 4 + 7 + 6 & & \multicolumn{1}{|l}{1 - 2} \\
\underline{1 - 2} & & \multicolumn{1}{|l}{\overline{1 + 6 + 19}} \\
\quad 6 + 7 & & = x^2 + 6x + 19 \\
\quad \underline{6 - 12} & & \\
\qquad 19 + 6 & & \\
\qquad \underline{19 - 38} & & \\
\qquad\quad 44, \text{ remainder} & &
\end{array}
$$

24. $x^3 + 3x^2 - 4x + 1$ by $x + 2$.

25. $a^4 + 2a^3 + 3a^2 + a - 2$ by $a - 1$.

26. $21x^4 + 4 - 8x^2 + 6x - 29x^3$ by $3x - 2$.

27. $y^4 + 7y - 10y^2 - y^3 + 15$ by $y^2 - 2y - 3$.

28. $7x^3 + 2x^4 - 27x^2 + 16 - 8x$ by $x^2 + 5x - 4$.

29. $a^5 - \frac{5}{4}a^4 + \frac{29}{24}a^3 - \frac{31}{24}a^2 + \frac{5}{6}a - \frac{1}{4}$ by $a - \frac{3}{4}$.

30. $a^4 - 1$ by $a - 1$.

31. $x^5 + 1$ by $x - 1$.

32. $x^6 - \frac{1}{64}$ by $x + \frac{1}{2}$.

33. $x^5 - 5x + 4$ by $x^2 - 2x + 1$.

34. $x^6 + x^3 + x + 1$ by $x^3 - x^2 + 1$.

35. $x^8 + 8x + 7$ by $x^2 + 2x + 1$.

36. Synthetic division. — This is an abridgment of the method of division by detached coefficients that is most useful and easily applied when the divisor is a binomial of the form $x \pm a$. For example, in the process at the top of the page, since the first number in each partial product is the same as the number directly above, it may be omitted and the terms of the dividend need not be brought down. The process may then be written more briefly thus:

$$
\begin{array}{lll}
1 + 4 + 7 + 6 & & \multicolumn{1}{|l}{1 - 2} \\
\quad \underline{-2} & & \multicolumn{1}{|l}{\overline{1 + 6 + 19}} \\
\quad\; 6 & & \\
\qquad \underline{-12} & & \\
\qquad\; 19 & & \\
\qquad\quad \underline{-38} & & \\
\qquad\qquad 44, \text{ remainder} & &
\end{array}
$$

We may write this process more compactly and further shorten the work by omitting the first term of the divisor and writing the second term *with its sign changed*, which will give all the partial products with *changed signs* so that we may *add* them to the dividend instead of subtracting them. Also, since each partial product now consists of but one term, we may write all the partial products in the same horizontal line under the dividend, thus:

$$\begin{array}{ll} \text{Dividend} & 1+4+\ 7+\ 6\ \ \underline{|2} \\ \text{Partial products} & \underline{\quad\ 2+12+38} \\ \text{Quotient} & 1+6+19\ |\ 44,\ \text{remainder} \end{array}$$

That is, the quotient is $x^2+6x+19$ and the remainder, 44.

EXERCISES

37. Divide by synthetic division:

1. $x^4+x^3-3x^2-17x-30$ by $x-3$; by $x+2$.

SOLUTIONS

$$\begin{array}{l} 1+1-\ 3-17-30\ \underline{|3} \\ \underline{\quad\ \ 3+12+27+30} \\ 1+4+\ 9+10 \end{array} \qquad \begin{array}{l} 1+1-3-17-30\ \underline{|-2} \\ \underline{\quad -2+2+\ 2+30} \\ 1-1-1-15 \end{array}$$

In the first case the quotient is $x^3+4x^2+9x+10$ and in the second it is x^3-x^2-x-15. The division is exact in both cases.

2. x^3+4x^2+5x+2 by $x+1$.

3. $1+2x+3x^2+4x^3$ by $1+x$.

4. $5v^3-12v+3v^2+4$ by $v+2$.

5. $y^4+3y^2-4y^3+8y-24$ by $y-3$.

6. $5a^5+2a^4-a^3-a^2+2a+3$ by $a-1$.

7. $x^7-x^6-2x^5-x^3+3x^4-10x+4x^2-36$ by $x-2$.

8. $t^5-2t^4+\frac{1}{12}t^3+\frac{2}{5}t^2+\frac{1}{15}t+\frac{5}{4}$ by $t-\frac{3}{2}$.

9. a^3+1 by $a+1$.

10. z^5-32 by $z-2$.

11. a^4-256 by $a+4$.

12. u^5+243 by $u+3$.

13. x^4-3x^2-4 by $x-2$.

14. $4y^4-3y^3-1$ by $y-1$.

15. $m^5-19m-6$ by $m+2$.

16. $a^6-38a+12$ by $a-2$.

Special Cases in Division

38. Show the truth of these *divisibility principles* for positive integral values of n, by substituting such values and actually dividing:

PRINCIPLES. — 1. *$x^n - y^n$ is always divisible by $x - y$.*

2. *$x^n - y^n$ is divisible by $x + y$ only when n is even.*

3. *$x^n + y^n$ is never divisible by $x - y$.*

4. *$x^n + y^n$ is divisible by $x + y$ only when n is odd.*

Proofs of these principles are given on page 54.

39. The following **law of signs** may be inferred readily:

When $x - y$ is the divisor, the signs in the quotient are plus.

When $x + y$ is the divisor, the signs in the quotient are alternately plus and minus.

40. The following **law of exponents** also may be inferred:

The quotient is ***homogeneous****, the exponent of x decreasing and that of y increasing by 1 in each successive term.*

EXERCISES

41. Write out the quotients by inspection:

1. $(c^3 + d^3) \div (c + d)$.

2. $(a^4 - b^4) \div (a - b)$.

3. $(r^5 + s^5) \div (r + s)$.

4. $(1 + a^5) \div (1 + a)$.

5. $(x^5 - y^5) \div (x - y)$.

6. $(x^6 - 1) \div (x + 1)$.

7. $(x^6 - 64) \div (x + 2)$.

8. $(x^3y^3 + a^3) \div (xy + a)$.

9. $(m^7 + n^7) \div (m + n)$.

10. $(a^7 + 128) \div (a + 2)$.

11. $(y^3 - 1000) \div (y - 10)$.

12. $(x^{10} + y^5z^5) \div (x^2 + yz)$.

13. By Prin. 4, find an exact binomial divisor of $a^6 + x^6$.

SUGGESTION. $a^6 + x^6$ may be written as the sum of two cubes thus, $(a^2)^3 + (x^2)^3$.

Find exact binomial divisors:

14. $a^3 - m^3$.

15. $b^3 + x^3$.

16. $x^5 - a^5$

17. $c^5 + n^5$.

18. $x^6 + y^6$.

19. $x^7 + a^7$.

20. $a^{10} + b^{10}$.

21. $a^3 - 27$.

22. $a^4 - b^4$, four.

23. $a^6 - 1$, five.

24. $a^8 - b^8$, six.

25. $a^{10} - b^{10}$, five.

EQUATIONS AND PROBLEMS

42. Write an **equation** and point out its **first member**; its **second member**.

43. The following **axioms** are constantly used in the **solution** of equations and problems:

1. *If equals are added to equals, the sums are equal.*
2. *If equals are subtracted from equals, the remainders are equal.*
3. *If equals are multiplied by equals, the products are equal.*
4. *If equals are divided by equals, the quotients are equal.*
5. *Numbers that are equal to the same number, or to equal numbers, are equal to each other.*
6. *The same powers of equal numbers are equal.*
7. *The same roots of equal numbers are equal.*

In the application of axiom 4, it is not allowable to divide by zero, or any number equal to zero, for the result cannot be determined (§ 283).

EXERCISES

44. **1.** Solve $x - 2 = 3$ by adding 2 to each member (Ax. 1).

2. Solve $x + 8 = 10$ by use of axiom 2.

3. Using axiom 3, find the value of x in $\frac{1}{4}x = 5$.

4. Apply axiom 4 to the solution of $5x = 30$.

5. Solve $\frac{3}{4}x = 12$ in two steps, first finding the value of $\frac{1}{4}x$ by axiom 4, and then the value of x by axiom 3.

Solve and give the axiom applying to each step:

6. $3x = 9$.
7. $4x = 6$.
8. $\frac{1}{2}x = 3$.
9. $\frac{1}{3}x = 5$.
10. $5x = 10$.
11. $3x = 16$.
12. $\frac{1}{5}x = 12$.
13. $\frac{1}{4}r = 1.5$.
14. $\frac{1}{2}x = 2.5$.
15. $y + 3 = 10$.
16. $v - 2 = 15$.
17. $4 + x = 20$.
18. $w - 4 = 16$.
19. $2z + 3 = 8$.
20. $\frac{2}{3}m = 8$.
21. $\frac{3}{4}w = 9$.
22. $\frac{3}{8}z = 15$.
23. $5n - 1 = 9$.
24. $4h + 3 = 5$.
25. $\frac{1}{2}b + 2 = 8$.
26. $\frac{1}{5}x + 2 = 6$.

Transposition in Equations

45. In solving the equations on page 33, the student may have discovered that the effect of applying axioms 1 and 2 has been to make a term disappear from one member of the equation and appear in the other member with its sign changed. That is,

PRINCIPLE. — *Any term may be* ***transposed*** *from one member of an equation to the other, provided its sign is changed.*

EXERCISES

46. 1. Solve $6x - 3(x - 6) = 4(2x - 1) + 2$, for x.

SOLUTION.	$6x - 3(x - 6) = 4(2x - 1) + 2.$
Expand,	$6x - 3x + 18 = 8x - 4 + 2.$
Transpose terms,	$6x - 3x - 8x = -18 - 4 + 2.$
Unite similar terms,	$-5x = -20.$
Divide both members by -5,	$x = 4.$

VERIFICATION. — Substituting 4 for x in the given equation, we have,

$$6 \cdot 4 - 3(4 - 6) = 4(2 \cdot 4 - 1) + 2.$$

Simplify each member, $30 = 30$, an **identity**.

Hence, 4 is a true value of x and **satisfies** the equation.

RULE. — *Remove signs of grouping, if there are any.*

Transpose terms so that the ***unknown numbers*** *stand in one member and the* ***known numbers*** *in the other.*

Unite similar terms and divide both members by the coefficient of the unknown number.

Find the value of x, and verify the result, in:

2. $3x - 4 = 5.$
3. $5 + \frac{1}{2}x = 8.$
4. $\frac{2}{3}x - 8 = -2.$
5. $2x + 6 = 8 - x.$
6. $1.5x - 7 = 5 + x.$
7. $24 - 2x = 3x - 6.$
8. $2x - \frac{1}{4}x = 6 + \frac{1}{4}x.$
9. $2(x - 1) = 12 - 5x.$
10. $16 = 5x - (3x + 1).$
11. $\frac{2}{3}x = 15 - \frac{1}{3}(x + 3).$
12. $3x = 5x - 4(x - 3).$
13. $5 - 2(x + 1) = 6 - 4x.$
14. $5(2 - x) + 6 = 2x - 5.$
15. $30 - x = 20 + 3(x + 2).$

Solve and verify:

16. $3x - 2(3 - x) = 9.$
17. $4(x - 2) = 3(x - 1).$
18. $3r = 2(1 - r) + 18.$
19. $5 - 3x - 7 + 6x = 0.$
20. $4s + 5 - \frac{3}{4}s = \frac{3}{4}s - 5.$
21. $(x - 3)(x + 2) = x^2 - 7.$
22. $2t(t - 5) - t^2 = t^2 + 30.$
23. $5x - 3(x - 4) = 4x + 7.$
24. $4(x - 5) - 3(x + 6) = 0.$
25. $x(x - 2) = (x - 3)^2 + 9.$
26. $3(2x - 4) = 4(x - 5) + 32.$
27. $3x - x^2 = x(1 - x) + 42.$

28. $(x + 1)^2 + (x + 3)^2 = 2(x^2 + 9).$
29. $\frac{1}{2}x + x(x - \frac{1}{2}) = x(x - 5) + 10.$
30. $(x + 2)^2 + (x - 3)^2 = 2(x + 4)^2 - 1.$
31. $5x - 24 + x^2 - 65 - 3(x - 2) = (x + 3)^2.$
32. $17x - (8x - 9) - [4 - 3x - (2x - 3)] = 30.$
33. $(x + 2)(x + 1)(x + 6) - 9x^2 = x^3 + 4(7x - 1).$
34. $(x + 1)(x - 1) - x^2 + (2x + 1)^2 = 4(x + 2)^2 + 8.$
35. $(x - 5)^2 + 2[3x - (x + 2)^2 + 5] = 3(x + 4) - x^2.$
36. $3[2x + 5 - 2\{x - 6 + 5x\} + x^2] = x(3x - 13).$
37. $(2x - 3)^2 - 4(1 - x)^2 = 2[x + 6 - 3(x - 8 + 4)] - x.$
38. $(4 - x)^2 - 2[8 - (x + 1)^2 - 3x] = 3(4 - x)^2 - 4(1 - 3x).$

Solve for x:

39. $ax + 16 = a^2 - 4x.$
40. $dx + 9a^2 = d^2 - 3ax.$
41. $x(1 - 3c) + 9c^2 = 1.$
42. $cx - 9 = c^2 + 6c - 3x.$
43. $ax - a^2 = 2ab + b^2 - bx.$
44. $3x - 12a = 4a^2 - 2ax + 9.$
45. $a(x + a) + b(b - x) = 2ab.$
46. $ax - c^2 = a^3 + ac + a^2c - cx.$

47. $a(x - a) - 2ab = -b(x - b).$
48. $(a^2 + x)(b^2 + x) = (ab + x)^2 + (a^2 - b^2)^2.$
49. $4m^4 - 2m^2x - 3mx = 1 - 6m + 9m^2 - x.$
50. $a^3 + a^2x - c^2x + x(c^2 - 1) + 2ax + x = x(a + 1)^2 - b^3.$
51. $c(2x - d) + c^2(d^2 - c) + d(x + c) = d^2(d + c) - c(d^2 - x) + c^2d^2.$

Problems

47. General Directions for Solving Problems. — 1. *Represent one of the unknown numbers by some letter, as x.*

2. *From the conditions of the problem find an expression for each of the other unknown numbers.*

3. *Find from the conditions two expressions that are equal and write the* ***equation of the problem.***

4. *Solve the equation.*

Solve each of the following problems:

1. What number multiplied by 3 is equal to 54?

SUGGESTION. — The equation of the problem is $3x = 54$.

2. What number increased by 10 is equal to 19?

3. What number diminished by 30 is equal to 20?

4. What number decreased by 6 gives a remainder of 17?

5. What number divided by 4 is equal to 24?

6. What number exceeds $\frac{1}{3}$ of itself by 10?

7. What number diminished by 45 is equal to -15?

8. What number is 3 more than $\frac{1}{2}$ of itself?

9. If $\frac{5}{6}$ of a number is 30, what is the number?

10. Find three consecutive numbers whose sum is 42.

11. Find three consecutive odd numbers whose sum is 57.

12. Find a number which added to its double equals 12.

13. Find three consecutive even numbers whose sum is 84.

14. Separate 64 into two parts whose difference is 12.

SUGGESTION. — Let x equal one part and $64 - x$, the other.

15. Separate 40 into two parts, one of which is 3 times the other.

16. If c times a number is $a + b$, what is the number?

17. If $\frac{2}{3}$ of a number is added to the number, the sum is 30. Find the number.

18. If $\frac{1}{3}$ of a number is added to twice the number, the sum is 35. What is the number?

19. The sum of two numbers is 35 and one number is $\frac{1}{4}$ of the other. Find the numbers.

20. If 5 times a certain number is decreased by 12, the remainder is 13. What is the number?

21. Eighty decreased by 7 times a number is 17. Find the number.

22. If I subtract 12 from 16 times a number, the result is 84. Find the number.

23. If from 7 times a number I take 5 times the number, the result is 18. What is the number?

24. One number is 8 times another; their difference is $14a$. What are the numbers?

25. The sum of a number and .04 of itself is 46.8. What is the number?

26. What number decreased by .35 of itself equals 52?

27. Find two numbers whose sum is 60 and whose difference is 36.

28. The sum of two numbers is 82. The larger exceeds the smaller by 16. Find the numbers.

29. Separate $2a$ into two parts, one of which is 4 more than the other.

30. Four times a certain number plus 3 times the number minus 6 times the number equals 7. What is the number?

31. If 5 times a certain number is subtracted from 58, the result is 16 plus the number. Find the number.

32. Twelve times a certain number is decreased by 4. The result is 6 more than 10 times the number. Find the number.

33. Three times a certain number decreased by 4 exceeds the number by 20. Find the number.

34. Three times a certain number is as much less than 72 as 4 times the number exceeds 12. What is the number?

35. Twice a certain number exceeds $\frac{1}{3}$ of the number as much as 6 times the number exceeds 65. What is the number?

36. Two boys had 350 apples. They sold the green ones for 3 ¢ each and the red ones for 5 ¢ each and received in all $ 11.60. How many apples of each kind did they sell?

SOLUTION. — Let $x =$ the number of green apples.

Then, $350 - x =$ the number of red apples,

$3x =$ the number of cents received for green apples,

and $5(350 - x) =$ the number of cents received for red apples.

$\therefore 3x+5(350-x) = 1160.$

Solving, we have $x = 295$, the number of green apples,

and $350 - x = 55$, the number of red apples.

VERIFICATION. — This solution satisfies the first condition of the problem; namely, the boys had 350 apples, for (295 + 55) apples = 350 apples. It also satisfies the second condition, for 295 × 3 ¢ + 55 × 5 ¢ = 1160 ¢, or $ 11.60. Hence, the solution is presumably correct.

Solve the following problems, and verify the solutions:

37. John and Frank have $ 72. John has $ 12 more than Frank. How many dollars has each?

38. Charles solved 14 problems, or $\frac{7}{8}$ of the problems in his lesson. How many problems were there in his lesson?

39. A house and lot cost $ 3000. If the house cost 4 times as much as the lot, what was the cost of each?

40. What is the number of feet in the width of a street, if $\frac{3}{5}$ of the width, or 48 feet, lies between the curbstones?

41. How long is one side of a square, if the perimeter added to the length of one side is 15 inches?

42. A and B began business with a capital of $ 7500. If A furnished half as much capital as B, how much capital did each furnish?

43. Ada is $\frac{3}{4}$ as old as her brother. If the sum of their ages is 28 years, how old is each?

44. If $\frac{3}{8}$ of the number of persons who went on an excursion to Niagara Falls were teachers, and 240 teachers went, what was the whole number of persons who went on the excursion?

45. I owe A and B \$45. If I owe A $\frac{4}{5}$ as much as I owe B, how much do I owe each?

46. A rectangle having a perimeter of 46 feet is 5 feet longer than it is wide. Find its dimensions.

47. Twelve years ago a boy was $\frac{1}{3}$ as old as he is now. What is his present age?

48. In 2 years A will be twice as old as he was 2 years ago. How old is he?

49. A lawn is 7 rods longer than it is wide. If the distance around it is 62 rods, what are its dimensions?

50. In a fire B lost twice as much as A, and C lost 3 times as much as A. If their combined loss was \$6000, how much did each lose?

51. A father is 4 times as old as his son. Six years ago he was 7 times as old as his son. Find the age of each.

52. In a business enterprise the joint capital of A, B, and C was \$8400. If A's capital was twice B's, and B's was twice C's, what was the capital of each?

53. How old is a man whose age 16 years hence will be 4 years less than twice his present age?

54. A boy is 8 years younger than his sister. In 4 years the sum of their ages will be 26 years. How old is each?

55. A prime dark sea-otter skin cost \$400 more than a brown one. If the first cost 3 times as much as the second, how much did each cost?

56. In 510 bushels of grain there was 4 times as much corn as wheat and 3 times as much barley as corn. How many bushels of each kind were there?

57. In a certain election at which 8000 votes were polled for A and B, B received 500 votes more than $\frac{1}{2}$ as many as A. How many votes did each receive?

58. A had \$ 40 more than B; B had \$ 10 more than $\frac{1}{3}$ as much as A. How much money had each?

59. A man has \$ 1.80. He has twice as many quarters as dimes. How many coins has he of each denomination?

60. A wagon loaded with coal weighed 4200 pounds. The coal weighed 1800 pounds more than the wagon. How much did the wagon weigh? the coal?

61. Mary bought 17 apples for 61 cents. For a certain number of them she paid 5 cents each, and for the rest she paid 3 cents each. How many of each kind did she buy?

62. A mining company sold copper ore at \$ 5.28 per ton. The profit per ton was \$ .22 less than the cost. What was the profit on each ton?

63. The students of a school numbering 210 raised \$ 175 with which to buy pictures. The seniors gave \$ 1.50 each, the rest \$ .50 each. Find the number of seniors.

64. A man has \$ 27.50 in quarters and half dollars, having 5 times as many half dollars as quarters. How many coins of each kind has he?

65. Two boys sold 150 tickets, the reserved seat tickets at 75 ¢ each and the others at 50 ¢ each. The total receipts were \$ 87.50. How many tickets of each kind did they sell?

66. The length of a classroom is 4 feet more than twice its width. If its width is increased 2 feet, the distance around it will be 120 feet. Find its dimensions.

67. My house is 16 feet deeper than it is wide. If it were 6 feet deeper than it is, the distance around it would be 140 feet. Find its dimensions.

68. A had 3 times as many marbles as B. A gave B 50 marbles; then B had twice as many as A. How many marbles had each?

FACTORS AND MULTIPLES

48. Review definitions and tell the meaning of:

1. **Factor.**
2. **Prime factor.**
3. **Factoring.**
4. **Prime to each other.**
5. **Rational expression.**
6. **Integral expression.**
7. **Degree of a term.**
8. **Degree of an expression.**
9. **Common factor.**
10. **Common multiple.**
11. **Highest common factor (h. c. f.).**
12. **Lowest common multiple (l. c. m.).**

FACTORING

49. Until noted farther on, the term *factor* will be understood to mean *rational integral factor*.

Monomial Factors

50. The *type form* is $nx + ny + nz = n(x + y + z)$, in which the terms of the expression have a common factor.

EXERCISES

51. Factor:

1. $3a^2x - 6ax^2 + 9ax$.

Solution. $3a^2x - 6ax^2 + 9ax = 3ax(a - 2x + 3)$.

2. $3b^2 + 3b^3$.
3. $2y^5 - 8y^4$.
4. $6a^2 + 4ab$.
5. $3xy^2 - 6x^2y^4$.
6. $2a^2 + 4a^3 + 6a^4$.
7. $5r^4 - 10r^2s + 5r^2s^2$.
8. $4a^2x^4 - 8a^3x^3 + 6a^4x^2$.
9. $6m^5n^3 + 9m^3n^5 - 3m^4n^4$.
10. $8ab^2c^2 - 4a^2b^3c^4 + 12a^3b^4c^6$.
11. $18r^2st^3 + 12rs^2t^4 - 24r^3st^2$.
12. $20b^2cd^3 - 15b^3c^2d + 25b^4cd^4$.
13. $9x^3y^3z^2 + 27x^2y^4z^3 - 18x^4y^2z^4$.

Factoring Binomials

52. Difference of two squares. The reverse of formula 3 (§ 30) gives the *type form*, $\boldsymbol{a^2 - b^2 = (a+b)(a-b)}$.

RULE. — *Find the square root of the two terms, and make their sum one factor and their difference the other factor.*

EXERCISES

53. Factor, and test each result:

1. $18c^2 - 50$; $x^4 - y^4$.

SOLUTIONS. $18c^2 - 50 = 2(9c^2 - 25) = 2(3c+5)(3c-5)$.

$$x^4 - y^4 = (x^2 + y^2)(x^2 - y^2) = (x^2 + y^2)(x+y)(x-y).$$

TEST. — The product of the factors should equal the given expression.

NOTE. — As in the above, sometimes the factors first found may be factored. When told to factor an expression, find its prime factors.

2. $9 - c^2$. **7.** $a^4 - 81$. **12.** $2x^4 - 2y^4$.

3. $a^4 - b^2$. **8.** $9b^2 - c^2d^2$. **13.** $5x^8 - 5y^8$.

4. $x^2 - 16$. **9.** $4x^2 - 25y^2$. **14.** $x^{2n+1} - xy^{2n}$.

5. $25 - c^2$. **10.** $9a^2 - 49b^2$. **15.** $9b^2 - (a-x)^2$.

6. $a^2x^2 - 1$. **11.** $16a^4 - 81b^4$. **16.** $(x^3 + x^2)^2 - (x+2)^2$.

54. Sum or difference of two cubes. Applying the principles of §§ 38–40, and taking the divisor and quotient for factors gives the *type forms*, $\boldsymbol{a^3 + b^3 = (a+b)(a^2 - ab + b^2)}$,

and $\boldsymbol{a^3 - b^3 = (a-b)(a^2 + ab + b^2)}$.

EXERCISES

55. Factor and test:

1. $c^3 + d^3$. **2.** $b^3 - c^3$. **3.** $x^3 + 8$. **4.** $y^3 - 125$.

5. $r^6 + s^6$; $a^9 - 125b^3$.

SOLUTIONS. $r^6 + s^6 = (r^2)^3 + (s^2)^3 = (r^2 + s^2)(r^4 - r^2s^2 + s^4)$.

$$a^9 - 125b^3 = (a^3)^3 - (5b)^3 = (a^3 - 5b)(a^6 + 5a^3b + 25b^2).$$

6. $a^6 + y^6$. **10.** $v^7 + 27v$. **14.** $1 + (a+b)^6$.

7. $x - x^4$. **11.** $r^6 + 64s^3$. **15.** $216x^{6n} + 64y^{3n}$.

8. $a^3b^3 - c^3$. **12.** $(x-y)^3 - 8$. **16.** $8(m+n)^3 + 125n^3$.

9. $x^{3n} + 64$. **13.** $r^{3x} - 729s^{3z}$. **17.** $(x-y)^3 - (x+y)^3$.

56. Sum or difference of the same odd powers. Applying principles §§ 38–40, as in § 54, gives for fifth powers the *type forms*,

$$a^5 + b^5 = (a + b)(a^4 - a^3b + a^2b^2 - ab^3 + b^4),$$

and $$a^5 - b^5 = (a - b)(a^4 + a^3b + a^2b^2 + ab^3 + b^4).$$

EXERCISES

57. Factor:

1. $m^5 + 32x^5$; $128a^{14} - 1$.

SOLUTIONS. $m^5 + 32x^5 = m^5 + (2x)^5$

$$= (m + 2x)(m^4 - 2m^3x + 4m^2x^2 - 8mx^3 + 16x^4).$$

$128a^{14} - 1 = (2a^2)^7 - 1$

$$= (2a^2 - 1)(64a^{12} + 32a^{10} + 16a^8 + 8a^6 + 4a^4 + 2a^2 + 1).$$

2. $x^5 + y^5$.

3. $x^5 - y^5$.

4. $a^7 - 1$.

5. $x^9 + y^9$.

6. $1 + y^7$.

7. $x^6 - x$.

8. $a^7 + 128$.

9. $64 - 2a^5$.

10. $a^6b - ab^6$.

11. $x^{5m} - y^{5n}$.

12. $m^7 - m^2$.

13. $a^9 + 512$.

14. $32 + a^{10}b^{10}$.

15. $1 - a^5b^{10}c^{15}$.

16. $x^{10} + 243a^5$.

58. Difference of the same even powers. Solve as in exercise 1, § 53.

MISCELLANEOUS EXERCISES

59. Factor each of these binomials, and test the result:

1. $x^2 - y^2$.

2. $r^3 + s^3$.

3. $y^2 - 9$.

4. $v^5 - 1$.

5. $n^3 + w^3$.

6. $z^3 - 27$.

7. $y^3 - 36y$.

8. $y^6 + a^3x^3$.

9. $1 - 144t^4$.

10. $128 - y^7z^7$.

11. $25x^2 - 1$.

12. $8x^6 - y^3$.

13. $a^7b^7 + 1$.

14. $a^{3n} - b^{3n}$.

15. $x^{16} - y^8$.

16. $3x^4 - 3y^6$.

17. $4a^2 - \frac{1}{9}$.

18. $x^{5a} + y^{5a}$.

19. $x^4 - \frac{1}{16}$.

20. $a^9 - x^9y^9$.

21. $x^{2n-2} - y^{4m}$.

22. $2x^8 - 2y^8$.

23. $243 + x^5y^5$.

24. $8a^6 - 729$.

25. $8a^{2n} - 18b^{2s}$.

26. $9a^2 - (2a - 5)^2$.

27. $27 + (x + y)^3$.

28. $x^{3n+1} - xy^{6n}$.

29. $(a - 2b)^2 - (a - 5)^2$.

30. $(r + s)^6 - 729t^6v^6$.

Factoring Trinomials

60. Trinomials that are perfect squares. Applying the reverses of formulas 1 and 2 (§ 30) gives the *type forms*,

$$a^2 + 2ab + b^2 = (a + b)^2,$$

and

$$a^2 - 2ab + b^2 = (a - b)^2.$$

It will be observed that these trinomials are **perfect squares**, for each is the product of two *equal* factors; also that a trinomial is a perfect square, if it has:

Two terms, as $+a^2$ and $+b^2$, that are perfect squares and another term that is numerically equal to twice the product of the square roots of the terms that are squares.

To factor a **trinomial square**:

RULE. — *Connect the square roots of the terms that are squares with the sign of the other term, and indicate that the result is to be taken twice as a factor.*

In factoring, usually only the *positive* square root is taken.
First remove the monomial factor, if there is one.

EXERCISES

61. Make a trinomial square by writing the missing term:

1. $x^2 + * + y^2$.
2. $a^2 - * + b^2$.
3. $y^2 + * + z^2$.
4. $c^2 - 2cd + *$.
5. $x^2 + 4xy + *$.
6. $r^2 - 8rs + *$.
7. $* + 4ab + b^2$.
8. $* - 2pq + q^2$.
9. $* + 6xy + y^2$.

Factor, and test each result:

10. $m^2 - 8m + 16$.
11. $4y - 4y^2 + y^3$.
12. $a^2 - 16a + 64$.
13. $3x^2 + 6xy + 3y^2$.
14. $9x^2 - 42x + 49$.
15. $36n^8 - 12n^4 + 1$.
16. $x^{2r} + 8x^rz + 16z^2$.
17. $2a^4 - 4a^2b^2 + 2b^4$.
18. $2x + 20a^2x + 50a^4x$.
19. $4x^{2a} + 8x^ay^b + 4y^{2b}$.
20. $8a^2b + 40ab^2 + 50b^3$.
21. $x^{2n} - 2x^ny^nz^n + y^{2n}z^{2n}$.
22. $x^2 + 2x(x - y) + (x - y)^2$.
23. $t^2 - 4t(t - 1) + 4(t - 1)^2$.
24. $14(x - y) + (x - y)^2 + 49$.
25. $c^2 - 6c(a - c) + 9(a - c)^2$.

62. Trinomials of the form $x^2 + px + q$. Applying the reverse of formula 4 (§ 30) gives,

$$x^2 + (a + b)x + ab = (x + a)(x + b),$$

which is in the *type form*, $x^2 + px + q$, having an x^2 term, an x term, and an **absolute term.**

Hence, if a trinomial of this form is factorable, it may be factored as follows:

RULE. — *Find two factors of q (the absolute term) such that their sum is p (the coefficient of x), and add each factor of q to x.*

EXERCISES

63. 1. Find the two binomial factors of $x^2 + 4x - 21$.

SOLUTION. — The first term of each factor is, of course, x.

The second terms of the factors must be two monomials whose algebraic sum is $+4$ and whose product is -21. Evidently these numbers must have *unlike* signs and it is seen that $+7$ and -3 fulfill the necessary conditions.

Hence, $\quad x^2 + 4x - 21 = (x + 7)(x - 3).$

Factor, and test each result:

2. $a^2 + 6a + 8$.
3. $x^2 + 3x - 10$.
4. $r^2 - 2r - 15$.
5. $y^2 + 7y - 18$.
6. $x^2 + 12x + 20$.
7. $c^2 + c - 30$.
8. $d^2 + 7d - 60$.
9. $2b^4 - 6b^2 - 56$.
10. $a^{2n} + 10a^n + 16$.
11. $x^{2n+1} + 3x^{n+1} + 2x$.
12. Factor $15 - n^2 + 2n$.

SUGGESTION. $\quad 15 - n^2 + 2n = -n^2 + 2n + 15 = -(n^2 - 2n - 15)$.

13. $12 + 4y - y^2$.
14. $z^2 + 18z + 56$.
15. $x^2 - abx - 2a^2b^2$.
16. $-t^2 - 16t + 36$.
17. $m^2 - 2mn - 15n^2$.
18. $-a^2 - 9a + 52$.
19. $x^2 - (c + d)x + cd$.
20. $x^2 - (a - d)x - ad$.
21. $2y^4 + 26y^2 - 180$.
22. $54a^2 - 3ay - y^2$.
23. $4ax - 2ax^2 + 48a$.
24. $3a^2 - 15ab - 72b^2$.
25. $x^2 - 2(a - n)x - 4an$.
26. $9b^2x + 54bx - 144x$.
27. $150n - 5nx^2 - 65nx$.
28. $20bx + 10b^2 - 630x^2$.

64. Trinomials of the general form $ax^2 + bx + c$. The types of trinomials so far treated are really special forms of the general type.

If the general quadratic trinomial $ax^2 + bx + c$ has binomial factors, they are of the forms $rx + t$ and $sx + v$.

EXERCISES

65. **1.** Factor $2x^2 - 5x - 3$.

SOLUTION. — If this trinomial is the product of two binomial factors, $2x^2$ is the product of their *first terms* and these terms must be $2x$ and x. Since -3 is the product of the last terms, they must have *unlike signs* and the only possible last terms are 3 and -1 or -3 and 1.

These first and last terms associated in all possible ways give:

$$\begin{array}{cccc} 2x-3 & 2x-1 & 2x+3 & 2x+1 \\ \underline{\ \ x+1\ \ } & \underline{\ \ x+3\ \ } & \underline{\ \ x-1\ \ } & \underline{\ \ x-3\ \ } \end{array}$$

Of these we select *by trial* the pair that will give $-5x$ (the middle term of the given trinomial) for the algebraic sum of the cross-products.

Hence, $$2x^2 - 5x - 3 = (2x + 1)(x - 3).$$

Observe that:

1. *When the sign of the last term of the trinomial is +, the last terms of the factors must be both + or both −, and like the sign of the middle term of the trinomial.*

2. *When the sign of the last term of the trinomial is −, the sign of the last term of one factor must be +, and of the other −.*

Factor, and test each result:

2. $2x^2 - 5x + 3$.
3. $3x^2 - 8x - 3$.
4. $2x^2 + 7x - 4$.
5. $6x^2 - 13x + 6$.
6. $5x^2 - 13x - 6$.
7. $8x^2 + 10x - 3$.
8. $15x^2 - 9x - 6$.
9. $6x^2 + 11x - 10$.
10. $27b^4 - 3b^2 - 14$.
11. $21a^2 - a - 10$.
12. $6x^2 - 10x + 4$.
13. $15x^2 + 22x + 8$.
14. $10x^4 - 11x^2 - 6$.
15. $16x^2 - 68x + 66$.
16. $10x^6 + 42x^3 + 44$.
17. $12x^2 + 14x - 40$.
18. $3x^2 + 7xy + 2y^2$.
19. $3x^2 + 5xy - 2y^2$.

When the coefficient of x^2 is a square, and when the square root of the coefficient of x^2 is contained exactly in the coefficient of x, the trinomial may be factored as follows:

20. Factor $9x^2 - 42x + 40$.

SOLUTION. $$\begin{aligned} 9x^2 - 42x + 40 &= (3x)^2 - 14(3x) + 40 \\ &= (3x - 4)(3x - 10). \end{aligned}$$

Factor, and test each result:

21. $4x^2 + 4x - 3$.

22. $9x^2 - 9x + 2$.

23. $4x^2 - 6x - 10$.

24. $9x^2 + 18x + 8$.

25. $16x^2 - 8x - 3$.

26. $25y^2 + 15y - 18$.

27. $36v^2 + 12v - 35$.

28. $32x^2 + 16x - 30$.

29. $49a^2 - 14a - 24$.

30. $81x^2 - 36x - 32$.

When the coefficient of x^2 is a square, and its square root is *not* contained exactly in the coefficient of x, *multiply and divide* by the coefficient of x^2, as follows:

31. Factor $4x^2 - 5x - 6$.

SOLUTION. $$\begin{aligned} 4x^2 - 5x - 6 &= (4x^2 - 5x - 6) \times \frac{4}{4} = \frac{16x^2 - 20x - 24}{4} \\ &= \frac{(4x)^2 - 5(4x) - 24}{4} = \frac{(4x - 8)(4x + 3)}{4} \\ &= \frac{4(x - 2)(4x + 3)}{4} = (x - 2)(4x + 3). \end{aligned}$$

32. Factor $24x^2 + 14x - 5$.

SUGGESTION. — *When the first term is not a square*, it may always be made a square whose square root will be contained exactly in the second term by *multiplying* the trinomial by the *coefficient of* x^2, or by a smaller multiplier. In this case multiply by 6, and divide by the same number.

Factor, and test each result:

33. $4x^2 + 19x - 5$.

34. $9y^2 - 13y - 10$.

35. $4a^2 + 17a - 15$.

36. $8x^2 + 22x + 9$.

37. $18b^2 + 46b - 24$.

38. $3x^2 - 10xy + 3y^2$.

39. $9x^2 + 43x - 10$.

40. $18x^2 - 9x - 35$.

41. $9x^4 - 10x^2 - 16$.

42. $16x^2 + 50x - 21$.

43. $32n^2 + 28n - 15$.

44. $5x^{2n} + 9x^ny - 2y^2$.

66. Trinomials of the form $a^4 + na^2b^2 + b^4$. By adding such a *positive perfect square* to the middle term as to make this trinomial a perfect square and then subtracting the same number so that the value will not be changed this *type form* becomes a special case of the difference of two squares (§ 52) whose type form is $a^2 - b^2$.

Thus, $a^4 + a^2b^2 + b^4 = a^4 + 2a^2b^2 + b^4 - a^2b^2 = (a^2 + b^2)^2 - a^2b^2$, whose factors are $a^2 + ab + b^2$ and $a^2 - ab + b^2$.

EXERCISES

67. 1. Factor $4x^4 - 13x^2 + 9$.

SOLUTION.
$$\begin{aligned} 4x^4 - 13x^2 + 9 &= 4x^4 - 12x^2 + 9 - x^2 \\ &= (2x^2 - 3)^2 - x^2 \\ &= (2x^2 + x - 3)(2x^2 - x - 3) \\ &= (2x + 3)(x - 1)(2x - 3)(x + 1). \end{aligned}$$

Factor, and test:

2. $x^4 + x^2y^2 + y^4$.
3. $b^4 + 3b^2 + 4$.
4. $x^8 + x^4z^4 + z^8$.
5. $9x^4 + 5x^2y^2 + y^4$.
6. $a^4 - 5a^2b^2 + 4b^4$.
7. $4y^4 + 7y^2z^2 + 4z^4$.
8. $a^4b^4 - 21a^2b^2 + 36$.
9. $9b^4 - 16b^2c^2 + 4c^4$.
10. $16c^4 - 17c^2d^2 + d^4$.
11. $y^4 - 37y^2z^2 + 36z^4$.
12. $9x^4 - 46x^2y^2 + 25y^4$.
13. $16a^4 + 15a^2b^2 + 9b^4$.
14. $25x^4 - 29x^2y^2 + 4y^4$.
15. $36a^4 - 52a^2b^2 + 16b^4$.

68. The method given in § 66 may be used to factor *binomials* of the *type form,* $\quad p^4 + 4$.

Thus, $p^4 + 4 = p^4 + 4p^2 + 4 - 4p^2 = (p^2 + 2)^2 - 4p^2$, whose factors are $p^2 + 2p + 2$ and $p^2 - 2p + 2$.

EXERCISES

69. Factor, and test:

1. $x^4 + 4$.
2. $4y^4 + 1$.
3. $2b^4 + 8$.
4. $m^8 + 4$.
5. $x^5 + 64x$.
6. $x^4 + 64y^4$.
7. $a^4 + 324$.
8. $2y^4 + 128$.
9. $4x^4 + 81y^4$.

MISCELLANEOUS EXERCISES

70. Factor orally each of these trinomials:

1. $a^2 + 4a + 4$.
2. $x^2 + 3x + 2$.
3. $z^2 + 5z + 4$.
4. $2x^2 - x - 1$.
5. $y^2 - 6y + 5$.
6. $9 + 6a^2 + a^4$.
7. $1 + 4s + 4s^2$.
8. $3x^2 + 6x + 3$.
9. $x^4 + x^2 + 1$.
10. $2y^2 + 3y - 2$.
11. $3v^2 - 8v - 3$.
12. $x^2 - 10x + 25$.
13. $x^2 + xy - 20y^2$.
14. $4x^2 + 8xy + 4y^2$.
15. $m^2 + 8mn + 16n^2$.
16. $m^2 - 6mn - 16n^2$.

Factor, and test each result:

17. $b^4 - 3b^2 - 4$.
18. $4x^2 + 8x + 3$.
19. $2y^2 - y - 15$.
20. $c^4 - 8c^2 + 16$.
21. $b^2 - 12b - 45$.
22. $5c^4 - 5c^2 - 60$.
23. $4x^2 - 5xy + y^2$.
24. $z^4 - 10z^2 + 24$.
25. $6x^2 - xy - 2y^2$.
26. $5x^2 - 38x + 21$.
27. $2x^2 + 5xy + 2y^2$.
28. $9x^2 - 27x + 18$.
29. $16 + 16a + 4a^2$.
30. $a^4x^4 + 3a^2x^2 - 28$.
31. $10a^3 + 14a^2 + 4a$.
32. $y^2 - (a - b)y - ab$.
33. $z^2 - (m + n)z + mn$.
34. $9a^2 + 12az^2 + 4z^4$.
35. $b^2 + 19bc + 48c^2$.
36. $36x^2 - 48x - 20$.
37. $4x^4 - 72x^2 + 324$.
38. $18x^2 - 51x + 36$.
39. $c^8d^8 + 7c^4d^4 + 12$.
40. $4a^8 - 48a^4 - 256$.
41. $16y^2 + 24yz - 7z^2$.
42. $4x^2 - 14xy + 10y^2$.
43. $25b^4 - b^2c^2 + 64c^4$.
44. $25y^2 - 25yz + 6z^2$.
45. $9b^2 + 49bc - 30c^2$.
46. $9b^4 - 13b^2x^2 + 4x^4$.
47. $4x^{6n} + 4x^{3n}y^n + y^{2n}$.
48. $49x^6 + 14x^3y - 15y^2$.
49. $(r + s)^2 - 4(r + s) + 4$.
50. $16 - 24(t - l) + 9(t - l)^2$.

Factoring Larger Polynomials

71. Polynomials whose terms may be grouped to show a common polynomial factor.

The *type form*, $\boldsymbol{ax + ay + bx + by}$,

may be solved as illustrated below.

Thus, $ax + ay + bx + by = a(x + y) + b(x + y) = (a + b)(x + y)$.

EXERCISES

72. 1. Factor $mx - my - nx + ny$.

SOLUTION.
$$\begin{aligned} mx - my - nx + ny &= (mx - my) - (nx - ny) \\ &= m(x - y) - n(x - y) \\ &= (m - n)(x - y). \end{aligned}$$

2. Factor $cx + y - dy + cy - dx + x$.

SOLUTION. $cx + y - dy + cy - dx + x$

Arranging terms,
$$\begin{aligned} &= cx - dx + x + cy - dy + y \\ &= x(c - d + 1) + y(c - d + 1) \\ &= (x + y)(c - d + 1). \end{aligned}$$

Factor, and test each result:

3. $bc + bx + cx + x^2$.
4. $ab + cx - ax - bc$.
5. $x^2 - xy + 3y - 3x$.
6. $bd - ac - bc + ad$.
7. $ax - by + bx - ay$.
8. $ax + 2y + 2x + ay$.
9. $2a + bx^2 + 2b + ax^2$.
10. $by - bx + 3ax - 3ay$.
11. $abc + acd + bd + a^2c^2$.
12. $bx - b^2y + abcy - acx$.
13. $6ab + 12b - 3ac - 6c$.
14. $5ax - 5ay + 3bx - 3by$.
15. $y^4 + y^3 + y + 1$.
16. $n^3 + n^2 - 4n - 4$.
17. $ay^2 - b + by^2 - a$.
18. $3m^2n - 9mn^2 + am - 3an$.
19. $36ab - 18ac - 18b^2 + 9bc$.
20. $15ab^2 - 9b^2c - 35ab + 21bc$.
21. $16ax + 12ay - 8bx - 6by$.
22. $ax^2 - ax - axy + ay + x - 1$.
23. $xy + x - 3y^3 - 3y^2 - 4y - 4$.
24. $mx - nx - x - my + ny + y$.
25. $bx^2 - b - xy - y + yx^2 - bx$.
26. $rx + sx + ry + sy + r + s$.

73. Polynomials as special cases of types a^2-b^2 and x^2+px+q. Many polynomials may be grouped as the difference of two squares.

EXERCISES

74. 1. Factor $a^2+2ab+b^2-1$.

SOLUTION. $a^2+2ab+b^2-1=(a^2+2ab+b^2)-1$

$=(a+b)^2-1$

Factor as in § 52, $=(a+b+1)(a+b-1)$.

2. Factor x^2-y^2-4x+4.

SUGGESTION. $x^2-y^2-4x+4=(x^2-4x+4)-y^2=(x-2)^2-y^2$.

3. Factor $a^2+b^2-c^2-4-2ab+4c$.

SOLUTION. $a^2+b^2-c^2-4-2ab+4c$

Arranging terms, $=a^2-2ab+b^2-c^2+4c-4$

$=(a^2-2ab+b^2)-(c^2-4c+4)$

$=(a-b)^2-(c-2)^2$

$=(a-b+c-2)(a-b-c+2)$.

Factor, and test each result:

4. $a^2+2a+1-b^2$.

5. $b^2-2bc+c^2-1$.

6. $1+2d+d^2-c^2$.

7. $a^2-b^2-2bd-d^2$.

8. $r^2-2rs+s^2-x^2$.

9. $n^2-x^2+2xy-y^2$.

10. $9c^2-y^2-z^2-2yz$.

11. $4a^2-c^2-d^2+2cd$.

12. $25y^4-1-4a-4a^2$.

13. $9x^2+6x+1-16a^2y^2$.

14. $bc^2-9a^2b-b^3-6ab^2$.

15. $ab^2-4a^3-12a^2c-9ac^2$.

16. $a^2-2ab+b^2-c^2+2cd-d^2$.

17. $x^2-2xy+y^2-m^2+10m-25$.

18. $a^2-4ab+4b^2-c^2-12c-36$.

19. $a^6+2a^5+a^4-a^2-2a-1$.

20. $x^2-a^2+y^2-b^2+2xy-2ab$.

21. $4x^2+9-12x+10mn-m^2-25n^2$.

Factor the following polynomials by writing them in the form $x^2 + px + q$, x^2 and x being replaced by polynomials.

22. Factor $9x^2 + 4y^2 + 12z^2 + 21xz + 14yz + 12xy$.

Solution. $\quad 9x^2 + 4y^2 + 12z^2 + 21xz + 14yz + 12xy$

Arranging terms, $= (9x^2 + 12xy + 4y^2) + (21xz + 14yz) + 12z^2$

$= (3x + 2y)^2 + 7z(3x + 2y) + 4z \cdot 3z$

§ 62, $= (3x + 2y + 4z)(3x + 2y + 3z)$.

23. $a^2 + 2ab + b^2 + 8ac + 8bc + 15c^2$.

24. $x^2 - 6xy + 9y^2 + 6xz - 18yz + 5z^2$.

25. $m^2 + n^2 - 2mn + 7mp - 7np - 30p^2$.

26. $9m^4 + k^2 - .0 + 39m^2 + 13k + 6m^2k$.

27. $16n^2 + 55 - 64n - 16m + m^2 + 8mn$.

28. $25a^2 + y^2 + 10x^2 + 10ay - 35ax - 7xy$.

75. Polynomials factorable for binomial factors by the factor theorem. If a product is equal to zero, at least one of the factors must be 0 or a number equal to 0.

Sometimes a polynomial in x reduces to 0 for more than one value of x. For example, $x^2 - 5x + 6$ equals 0 when $x = 3$ and also when $x = 2$; or when $x - 3 = 0$ and $x - 2 = 0$. In this case both $x - 3$ and $x - 2$ are factors of the polynomial.

76. Factor Theorem. — *If a polynomial in x, having positive integral exponents, reduces to zero when r is substituted for x, the polynomial is exactly divisible by $x - r$.*

The letter r represents any number that we may substitute for x.

Proof. — Let D represent any rational integral expression containing x, and let D reduce to zero when r is substituted for x.

It is to be proved that D is exactly divisible by $x - r$.

Suppose that the dividend D is divided by $x - r$ until the remainder does not contain x. Denote the remainder by R and the quotient by Q.

Then,
$$D = Q(x - r) + R. \qquad (1)$$

But, since D reduces to zero when $x = r$, that is, when $x - r = 0$, (1) becomes
$$0 = 0 + R; \text{ whence, } R = 0.$$

That is, the remainder is zero, and the division is exact.

EXERCISES

77. **1.** Factor $x^3 - x^2 - 9x + 9$.

Solution. — When $x = 1$, $x^3 - x^2 - 9x + 9 = 1 - 1 - 9 + 9 = 0$.

Therefore, $x - 1$ is a factor of the given polynomial.

Dividing $x^3 - x^2 - 9x + 9$ by $x - 1$ gives the quotient $x^2 - 9$.

By § 52, $\quad x^2 - 9 = (x + 3)(x - 3)$.

Hence, $\quad x^3 - x^2 - 9x + 9 = (x - 1)(x + 3)(x - 3)$.

Notes. — 1. Only factors of the *absolute term* of the polynomial need be substituted for x in seeking the binomial factors of the polynomial, for if $x - r$ is one factor, the absolute term of the polynomial is the product of r and the absolute term of the other factor.

2. Since when 1 is substituted for x the value of the polynomial is equal to the sum of its coefficients, $x - 1$ is a factor of a polynomial when the sum of its coefficients is equal to 0.

3. In testing for factors, instead of using ordinary substitution it is convenient to employ synthetic division, for this will show in one operation whether or not there is a remainder and give the quotient of the polynomial by the factor being tried. Thus, in exercise 1 in trying the factor $x - 1$, we have,

$$\begin{array}{l|l} 1 - 1 - 9 + 9 & \underline{1} \\ \quad + 1 - 0 - 9 & \\ \hline 1 + 0 - 9 & \end{array}$$

which shows at once that there is no remainder and the quotient is $x^2 - 9$.

16

2. Factor $2x^3 - 9x^2 - 2x + 24$.

Solution

Since the sum of the coefficients is not equal to 0, $x - 1$ is not a factor.

Using synthetic division to try for the factor $x - 2$, we have,

$$\begin{array}{l|l} 2 - 9 - \;\; 2 + 24 & \underline{2} \\ \quad + 4 - 10 - 24 & \\ \hline 2 - 5 - 12 & \end{array}$$

which shows that $x - 2$ is a factor and that when this factor is divided out the quotient is $2x^2 - 5x - 12$.

By § 64, $\quad 2x^2 - 5x - 12 = (x - 4)(2x + 3)$.

Hence, $\quad 2x^3 - 9x^2 - 2x + 24 = (x - 2)(x - 4)(2x + 3)$.

3. Factor $3x^3 - 8x^2y + 3xy^2 + 2y^3$.

Suggestion. — When $x = y$,

$$3x^3 - 8x^2y + 3xy^2 + 2y^3 = 3y^3 - 8y^3 + 3y^3 + 2y^3 = 0.$$

Therefore, $x - y$ is a factor of $3x^3 - 8x^2y + 3xy^2 + 2y^3$.

Factor by the factor theorem:

4. $x^3 + 4x^2 + x - 6$.
5. $x^3 + 2x^2 - 5x - 6$.
6. $x^3 + 6x^2 + 5x - 12$.
7. $x^3 - 7x^2 + 7x + 15$.
8. $x^3 - 12x^2 + 41x - 30$.
9. $a^3 + 4a^2 - 11a - 30$.
10. $x^3 - 13x^2 + 46x - 48$.
11. $a^3 + 9a^2 + 26a + 24$.
12. $2x^3 - 3x^2 - 17x - 12$.
13. $x^3 - 16x^2 + 71x - 56$.
14. $2x^3 - 9x^2 - 2x + 24$.
15. $2n^3 - 7n^2 - 7n + 30$.
16. $n^3 + 12n^2 + 41n + 42$.
17. $x^3 - 27x + 54$.
18. $x^3 - 39x - 70$.
19. $a^3 - 7ab^2 + 6b^3$.
20. $x^3 - 21xy^2 + 20y^3$.
21. $b^3 - 5b^2 - 29b + 105$.
22. $a^3 + 10a^2 - 17a - 66$.
23. $x^3 + 2x^2y - xy^2 - 2y^3$.
24. $x^3 + 4x^2y + 5xy^2 + 2y^3$.
25. $b^3 + 16b^2 + 73b + 90$.
26. $x^4 - 15x^2 + 10x + 24$.
27. $x^4 + 8x^3 + 14x^2 - 8x - 15$.
28. $x^5 - 2x^4 - 5x^3 + 14x + 12$.
29. $x^5 - 4x^4 + 19x^2 - 28x + 12$.

78. Proofs of divisibility principles for $x^n \pm y^n$. The principles laid down by experiment in § 38, and later used in factoring certain binomials may be proved readily by the factor theorem:

Proof of Prin. 1. — In $x^n - y^n$, substitute y for x; then, for any positive integral value of n, $x^n - y^n = y^n - y^n = 0$.

Hence, $x - y$ is a factor of $x^n - y^n$.

That is, *$x^n - y^n$ is always divisible by $x - y$.*

Proof of Prin. 2. — In $x^n - y^n$, substitute $-y$ for x; then, $x^n - y^n = (-y)^n - y^n$, which is equal to 0 when n is even but not when n is odd.

Hence, $x + y$ is a factor of $x^n - y^n$ only when n is even.

That is, *$x^n - y^n$ is divisible by $x + y$ only when n is even.*

Proof of Prin. 3. — In $x^n + y^n$, substitute y for x; then, for any positive integral values of n, $x^n + y^n = y^n + y^n$, which is not equal to 0.

Hence, $x - y$ is not a factor of $x^n + y^n$.

That is, *$x^n + y^n$ is never divisible by $x - y$.*

Proof of Prin. 4. — In $x^n + y^n$, substitute $-y$ for x; then, $x^n + y^n = (-y)^n + y^n$, which is equal to 0 when n is odd but not when n is even.

Hence, $x + y$ is a factor of $x^n + y^n$ only when n is odd.

That is, *$x^n + y^n$ is divisible by $x + y$ only when n is odd.*

Summary of Factoring

79. In the previous pages the student has learned to factor expressions of the following *types*.

MONOMIAL FACTORS

Common to all terms, $nx + ny + nz$.

BINOMIALS

$a^2 - b^2$.	$a^n \pm b^n$ (when n is odd).
$a^3 + b^3$.	$a^n - b^n$ (when n is even, as in $a^2 - b^2$).
$a^3 - b^3$.	$p^4 + 4$ (special case of $a^2 - b^2$).

TRINOMIALS

$a^2 \pm 2ab + b^2$.	$ax^2 + bx + c$.
$x^2 + px + q$.	$a^4 + na^2b^2 + b^4$ (special case of $a^2 - b^2$).

LARGER POLYNOMIALS

With common polynomial factor,	$ax + ay + bx + by$.
Special cases of types,	$a^2 - b^2$ and $x^2 + px + q$.
Having binomial factors,	**by the factor theorem.**

80. **General directions for factoring.** — 1. *Remove monomial factors, if there are any.*

2. *Determine whether the resulting expression is a binomial, a trinomial, or a larger polynomial, then decide to which type under that head it belongs, and factor by the proper method for that type.*

3. *Continue as in 2 with each factor found until the given expression is resolved into its prime factors.*

NOTE. — The factor theorem is applicable to binomials and trinomials as well as to larger polynomials and may often be used when other methods fail.

MISCELLANEOUS EXERCISES

81. Factor orally:

1. $2a - 2b.$
2. $x^2 - 3x.$
3. $x^2 - y^2.$
4. $a^2 - 1.$
5. $x^2 + x.$
6. $4a^2 + 4a.$
7. $8x^3 - 2x^2.$
8. $a^2 - b^2.$
9. $c^2 - 4.$
10. $x^3 + y^3.$
11. $y^3 - x^3.$
12. $x^2 - 9.$
13. $x^3 - x.$
14. $y^{2a} - 1.$
15. $x^{n+1} + x.$
16. $y^{2n+1} - y.$
17. $x^4 - y^4.$
18. $x^3 - 1.$
19. $a^3 - 8.$
20. $m^3 + 1.$
21. $a^2 - 4b^2.$
22. $d^3 - 9d.$
23. $a^3 - a^2y.$
24. $4y^4 - 4y.$
25. $x^2 - 2x + 1.$
26. $a^2x - 2a^2x^2.$
27. $x^2 + 3x + 2.$
28. $a^2 - (b + c)^2.$
29. $x^2 - x - 6.$
30. $x^2 + x - 6.$
31. $x^2 + x - 2.$
32. $x^2 - 2x - 3.$
33. $5a^2b^2 + 5a^4b^4.$
34. $x^2 + 5x + 6.$
35. $(x - y)^2 - z^2.$
36. $4a^2 - 9b^2.$
37. $x^{2a} - y^{2a}.$
38. $c^3 + 8d^3.$
39. $a^{4n} - b^2.$
40. $x^{2n+2} - 1.$
41. $x^2 + 2xy + y^2.$
42. $8x^4y^4 + 10x^3y^3.$
43. $x^2 + 3xy + 2y^2.$
44. $3 + 4x + x^2.$
45. $x^2 + ax + x + a.$
46. $x^2 - x + x - 1.$
47. $ab - bx + ac - cx.$
48. $x^3 + x^2 + x + 1.$

Factor, and test each result:

49. $x^6 + x.$
50. $b^7 - c^7.$
51. $r^8 - s^8.$
52. $y^6 - z^6.$
53. $1 - a^{10}.$
54. $a^9 - b^3.$
55. $a^6 + x^6.$
56. $x^{10} + 1.$
57. $a^4 - 256.$
58. $x^{3n} - a^{3b}.$
59. $z^6 + 32z.$
60. $5y^4 + 20.$
61. $a^{13} - ab^{12}.$
62. $7n^7 + 7n.$
63. $x^9 + 4x.$
64. $c^9 - 16c.$
65. $125 - 8x^6.$
66. $16m^3 + 2.$
67. $z^4 + z^2 + 1.$
68. $2x^2 + x - 1.$
69. $x^2 + 9x - 90.$
70. $1 + (x + 1)^3.$
71. $3x^2 - 2x - 8.$
72. $15 + 6x - 9x^2.$

73. $1-(x+1)^3$.

74. $1000x^3-27y^3$.

75. $a^4b^2+a^2b-12$.

76. $25p^{2m}-36x^{2p}$.

77. $17-16a-a^2$.

78. $6b^2-7b-3$.

79. $4a-3ax-ax^2$.

80. $12c^2+7c-12$.

81. x^2-9y^2+6x+9.

82. $3x^2+7xy-6y^2$.

83. $az-z+2a-2$.

84. $a^4x^4+2a^2x^2+9$.

85. $60a^2+8ax-3x^2$.

86. $9b^4+21b^2c^2+25c^4$.

87. $25y^2-25yz+6z^2$.

88. $10a^2c+33ac-7c$.

89. $(a+b)^6-1$.

90. $(a+x)^4-x^4$.

91. $17x^2+25x-18$.

92. $(x+y)^3+(x-y)^3$.

93. $(a-2)^3+(a-1)^3$.

94. $4x^3+x^2-8x-2$.

95. $x^2+5x+ax+5a$.

96. $x^4-119x^2y^2+y^4$.

97. $(a+b)^4-(b-c)^4$.

98. $3ab(a+b)+a^3+b^3$.

99. $(x^2-y^2)^2-(x^2-xy)^2$.

100. $(a^2+b^2-c^2)^2-4a^2b^2$.

101. $x^{2n-2}+b^2y^2+2x^{n-1}by$.

102. $ab-bx^n+x^ny^m-ay^m$.

103. $x^3+15x^2+75x+125$.

104. $a^4-b^4-(a+b)(a-b)$.

105. $x^2-z^2+y^2-a^2-2xy+2az$.

106. $2b^2m-3ab^2+2bmx-3abx$.

107. $a^2+b^2+c^2-2ab-2ac+2bc$.

108. $3ab^2x^2+4cdy-4ab^2xy-3cdx$.

109. $9x^2+y^2+16z^2-6xy-8yz+24xz$.

110. $x^2+9y^2+25z^2-6xy-10xz+30yz$.

111. $9a^2-12ab-4c^2-12cd+4b^2-9d^2$.

112. $x^2y^2z^2+a^2b^2+1+2abxyz+2xyz+2ab$.

113. Factor $32-x^5$ by the factor theorem.

114. If n is odd, factor x^n-a^n by the factor theorem.

115. If n is odd, factor x^n+r^n by the factor theorem.

116. Factor $x^3-9x^2y+27xy^2-27y^3$ by the factor theorem.

117. Discover by the factor theorem for what values of n between 1 and 20, x^n+a^n has no binomial factors.

EQUATIONS SOLVED BY FACTORING

EXERCISES

82. **1.** Solve the equation $x^2 + 1 = 2x + 16$.

SOLUTION. $x^2 + 1 = 2x + 16$.

Transpose all the terms to the first member and unite similar terms,

$$x^2 - 2x - 15 = 0.$$

Factor the first member, $(x - 5)(x + 3) = 0$.

If a product is equal to 0 at least one of its factors is equal to 0; that is,

$$x - 5 = 0 \text{ or } x + 3 = 0,$$

whence, $x = 5$ or $x = -3$.

VERIFICATION. — Substituting these values of x in the given equation, we find that each satisfies the equation.

Solve for x by factoring:

2. $x^2 - 1 = 3$.

3. $x^2 + 3 = 28$.

4. $x^2 + 35 = 39$.

5. $x^2 - 50 = 50$.

6. $x^2 - 4b^2 = 0$.

7. $x^2 - 9n^2 = 0$.

8. $x^2 - 40 = 24$.

9. $x^2 - 3a^2 = 6a^2$.

10. $x^2 + 5b^4 = 6b^4$.

11. $x^2 - 3x = 40$.

12. $x^2 - 9x = -20$.

13. $x^2 + 12x = 28$.

14. $x^2 - 10x = 96$.

15. $x^2 + 12x = 85$.

16. $600 = x^2 - 10x$.

17. $4x^2 - 8b^2 = 8b^2$.

18. $3x^2 + 11x = 4$.

19. $x^2 - a^2 = 2a + 1$.

20. $x^2 + 2bx + b^2 = 0$.

21. $2x^2 - 1 = 14 - x$.

22. $x^2 - b^4 = 4 - 4b^2$.

23. $3x^2 - 7x - 4 = 2$.

24. $x^2 - c^2 = d^2 - 2cd$.

25. $4x^2 + 9x - 9 = 0$.

26. $x^4 - 4x^3 + 2x^2 + 4x - 3 = 0$.

SUGGESTION. — Factor by the factor theorem.

27. $2x^4 - 5x^3 - 23x^2 + 36x + 28 = 4 - 2x$.

28. $x^5 - 10x^4 + 40x^3 - 80x^2 + 80x - 32 = 0$.

29. $3x^4 + 3x^3 - 47x^2 - 56x + 180 = 7x - 4x^3$.

HIGHEST COMMON FACTOR

83. PRINCIPLE. — *The highest common factor of two or more expressions is equal to the product of all their common prime factors.*

EXERCISES

84. Find the highest common factor of:

1. $c^2 - d^2$ and $c^2 - 2\,cd + d^2$.
2. $x^4 + xy^3$ and $x^2y + xy^2$.
3. $a^3 - b^3$ and $a^2 - 2\,ab + b^2$.
4. $x^2 - y^2$, $x^4 - y^4$, and $y^2 - x^2$.
5. $a^2 - x^2$, $a^2 + 2\,ax + x^2$, and $a^3 + x^3$.
6. $x^2 + 7\,x - 60$ and $x^2 - 12\,x + 35$.
7. $a^4 + a^2b^2 + b^4$ and $a^2 - ab + b^2$.
8. $1 - 4\,c^2$, $2\,a - 8\,ac^2$, and $2\,c - 1$.
9. $(a - b)(b - c)$ and $(c - a)(b^2 - a^2)$.
10. $16\,x^2 - 25$ and $20\,x^2 - 9\,x - 20$.
11. $5\,x^4 + 5\,x^2 + 5$ and $5\,ax^2 - 5\,ax + 5\,a$.
12. $x^7y + xy^4$ and $2\,x^5y - 2\,x^3y^2 + 2\,xy^3$.
13. $6\,x^2 - 5\,x - 6$ and $9\,x^2 - 6\,x - 8$.
14. $by - z + yz - b$ and $by^2 + y^2z - b - z$.
15. $6\,x^2 - 54$, $9(x + 3)$, and $30(x^2 - x - 12)$.
16. $8\,a - 8\,a^2$, $12\,a(a^2 - 1)^2$, and $18\,a^2 - 36\,a + 18$.
17. $9\,a^2(x^2 - 8\,x + 16)$ and $3\,a^2x + 6\,ax - 12\,a^2 - 24\,a$.
18. $x^2 - 4$ and $x^3 - 10\,x^2 + 31\,x - 30$.
19. $3\,x^4 - 12\,x^2$ and $6\,x^4 + 30\,x^3 - 96\,x^2 + 24\,x$.
20. $a^4b - a^2b^3$ and $a^4b + 2\,a^3b^2 + 2\,a^2b^3 + ab^4$.
21. $4 - a^2$ and $a^4 + a^3 - 10\,a^2 - 4\,a + 24$.
22. $(x - x^2)^3$, $(x^2 - 1)^3$, and $(1 - x)^3$.
23. $(1 - y^4)^2$ and $(y + 1)^2(1 - y)^2(y^3 - 7\,y + 6)$.
24. $x^2 - (y + z)^2$, $(y - x)^2 - z^2$, and $y^2 - (x - z)^2$.

LOWEST COMMON MULTIPLE

85. Principle. — *The lowest common multiple of two or more expressions is equal to the product of all their different prime factors, each factor being used the greatest number of times it occurs in any of the expressions.*

EXERCISES

86. Find the lowest common multiple of:

1. $a^2 - b^2$ and $a^2 + 2ab + b^2$.
2. $r^2 - s^2$ and $r^2 - 2rs + s^2$.
3. $2c^2d + 4cd^2 + 2d^3$ and $c^2 - d^2$.
4. $a^2 - b^2$ and $ax - a + bx - b$.
5. $x^2 + 5x + 6$ and $x^2 + 6x + 8$.
6. $a^2 - 5ab + 4b^2$ and $a^2 - 2ab + b^2$.
7. $x(a^3 - b^3)$, $x^2(a - b)$, and $a^2 + ab + b^2$.
8. $2a + 1$, $4a^2 - 1$, and $8a^3 + 1$.
9. $x^4 - 16$, $x^2 + 4x + 4$, and $x^2 - 4$.
10. $1 - x^2$, $x^2 + x$, $xy - y$, and $x^2 + 1$.
11. $3 + 3a$, $2a - 2$, $1 - a^2$, and $4 - 4a$.
12. $x^2 + 5x + 6$, $x^2 - x - 12$, and $x^2 - 2x - 8$.
13. $15(a^2b - ab^2)$, $21(a^3 - ab^2)$, and $35(ab^2 + b^3)$.
14. $x^2 - 8x + 15$, $x^2 - 4x - 5$, and $x^2 - 2x - 3$.
15. $xy - y^2$, $x^2 + xy$, $xy + y^2$, and $x^2 + y^2$.
16. $y^3 - x^3$, $x^2 + xy + y^2$, and $x^2 - xy$.
17. $m - n$, $(m^2 - n^2)^2$, and $(m + n)^3$.
18. $a^6 - b^3$ and $a^8 + a^4b^2 + b^4$.
19. $x^6 + y^6$ and $a^2x^2 - b^2y^2 + a^2y^2 - b^2x^2$.
20. $a^4 - a^2 + 1$, $a^6 + 1$, $a^4 + a^2 + 1$, and $a^2 - 1$.
21. $x^3 - 7x - 6$ and $x^3 - 2x^2 - 5x + 6$.
22. $x^3 - x^2 - 14x + 24$ and $x^3 - 3x^2 - 18x + 40$.
23. $x^3 + 5x^2 - 18x - 72$ and $x^4 + 2x^3 - 25x^2 - 26x + 120$.

FRACTIONS

87. State the difference between the *arithmetical* and *algebraic* notions of a fraction. Define **numerator** and **denominator.**

88. Signs in fractions. Operations with algebraic fractions are performed as in arithmetic except that the signs, of which there are three, must be considered. They are the sign of the numerator, the sign of the denominator, and the sign before the fraction.

By the law of signs for division (§ 32):

$\frac{-a}{-b}=+\frac{a}{b}$; $\frac{+a}{+b}=+\frac{a}{b}$; $\frac{-a}{+b}=-\frac{a}{b}$; and $\frac{+a}{-b}=-\frac{a}{b}$. That is,

PRINCIPLES. — 1. *The signs of both terms of a fraction may be changed without changing the sign of the fraction.*

2. *The sign of either term of a fraction may be changed, provided the sign of the fraction is changed.*

The sign of a polynomial numerator or denominator is changed by changing the sign of each of its terms.

By the law of signs for multiplication, *the sign of either term of a fraction is changed by changing the signs of an odd number of its factors, and left unchanged by changing the signs of an even number of its factors.*

EXERCISES

89. Reduce to fractions whose **terms** are positive:

1. $\frac{-2}{-3}$.
2. $\frac{-x}{y}$.
3. $\frac{b+c}{-a}$.
4. $\frac{-z}{-x-y}$.
5. $-\frac{r+s}{-u-v}$.
6. $-\frac{-a-b}{x+y}$.
7. $-\frac{-y-z}{-x-z}$.
8. $\frac{-a(b+c)}{b(-a-b)}$.

9. Show that $\frac{(b-a)(a-b+c)}{(a-b)(b-c)(c-d)}=\frac{(a-b)(a-b+c)}{(b-a)(c-b)(d-c)}$.

REDUCTION OF FRACTIONS

90. Define **reduction**; **mixed number**; **integral expression.**

Reduction to Integers or Mixed Numbers

EXERCISES

91. 1. Reduce $\frac{b^2z^3 - bz^2 - z - 1}{bz}$ to a mixed number.

SOLUTION. — Since a fraction is an indicated division, we simply divide the numerator by the denominator until the undivided part of the numerator no longer contains the denominator, thus:

$$\frac{b^2z^3 - bz^2 - z - 1}{bz} = bz^2 - z + \frac{-z-1}{bz} = bz^2 - z - \frac{z+1}{bz}.$$

Reduce to an integral or a mixed expression:

2. $\frac{35}{5}$.
3. $\frac{26}{3}$.
4. $\frac{6xy}{3x}$.
5. $\frac{8a^2bc^3}{2abc}$.
6. $\frac{2y+z}{2}$.
7. $\frac{ab+d}{a}$.
8. $\frac{xy+m}{y}$.
9. $\frac{4x^2+6x}{2x}$.
10. $\frac{abc+2ab}{ab}$.
11. $\frac{x^2+2x+1}{x}$.
12. $\frac{b^3+b^2+2}{b^2}$.
13. $\frac{x^2+x-12}{x^2}$.
14. $\frac{x^2-2x-8}{x-4}$.
15. $\frac{x^2-6x+5}{x-1}$.
16. $\frac{a^2+4ab+4b^2}{a+b}$.
17. $\frac{6x^2+19x+10}{3x+2}$.
18. $\frac{a^4+3a^2b^2+b^4}{a^2+b^2}$.
19. $\frac{4x^2+22x+21}{2x+4}$.
20. $\frac{x^3+2x^2+3x+1}{x+2}$.
21. $\frac{a^4-a^3b+2a^2b^2+b^4}{a^2+b^2}$.
22. $\frac{m^3-2m^2n-3mn^2-2n^3}{m^2-n^2}$.
23. $\frac{x^4+4x^3y+6x^2y^2+4xy^3}{x+y}$.

Reduction to Lowest Terms

92. What are **equivalent fractions**? When is a fraction said to be in its **lowest terms**?

93. PRINCIPLE. — *Multiplying or dividing both terms of a fraction by the same number does not change the value of the fraction.*

EXERCISES

94. **1.** Reduce $\frac{a^2-2ab+b^2}{b^2-a^2}$ to its lowest terms.

SOLUTION. $\frac{a^2-2ab+b^2}{b^2-a^2}=-\frac{(a-b)\cancel{(a-b)}}{(a+b)\cancel{(a-b)}}=-\frac{a-b}{a+b}.$

RULE. — *To reduce a fraction to its lowest terms, factor both terms, and divide both terms by their common factors.*

Use cancellation wherever possible.

Reduce to lowest terms:

2. $\frac{18}{24}$. **4.** $\frac{a^2bc^3}{ab^3c}$. **6.** $\frac{x^{m+2}y^2}{x^my^4}$. **8.** $\frac{4x^2}{2x^2+2xy}$.

3. $\frac{20}{32}$. **5.** $\frac{x^5y^4z}{x^3y^5z^2}$. **7.** $\frac{a^{m+2r}y^{2r}}{2a^ry^{4r}}$. **9.** $\frac{a^2b^2c-abc^2d}{abc}$.

10. $\frac{x^2-y^2}{x^2-2xy+y^2}$.

11. $\frac{3a^2+3ab}{a^4+ab^3}$.

12. $\frac{2x^2y^2-8y^4}{4x^3y-32y^4}$.

13. $\frac{10nx+10ny}{25nx^2-25ny^2}$.

14. $\frac{y^2-81}{y^2+7y-18}$.

15. $\frac{b^2+b-12}{3b^2+9b-54}$.

16. $\frac{(a+b)^2-1}{a^2c+abc+ac}$.

17. $\frac{x^{n+2}-x^n}{x^{n+3}-x^n}$.

18. $\frac{a^{n+4}-a^ny^4}{a^{n+3}+a^{n+1}y^2}$.

19. $\frac{a^2b(a+2b)^4}{ab(a^2-4b^2)^2}$.

20. $\frac{x^4+x^2y^2+y^4}{x^3+y^3}$.

21. $\frac{cd^3-c}{c^2d^4+c^2d^2+c^2}$.

22. $\frac{x^5-x^2-x^4+x^3}{x^4-1}$.

23. $\frac{a^3-ab-a^2b+b^2}{a^4-a^2b-a^2b^2+b^3}$.

Reduction to Lowest Common Denominator

95. When are fractions said to have a **common denominator**? their **lowest common denominator (l. c. d.)**?

EXERCISES

96. Reduce to respectively equivalent fractions having their lowest common denominator:

1. $\dfrac{a^2}{a^2-1}$, a, and $\dfrac{a}{a-1}$.

Solution. — Since the l. c. m. of the given denominators is a^2-1, each fraction or integer must be reduced to a fraction whose denominator is a^2-1.

Then, $\dfrac{a^2}{a^2-1}=\dfrac{a^2}{a^2-1}$; $a=\dfrac{a}{1}=\dfrac{a(a^2-1)}{a^2-1}$; and $\dfrac{a}{a-1}=\dfrac{a(a+1)}{a^2-1}$.

Rule. — *Find the lowest common multiple of the denominators of the fractions for the lowest common denominator.*

Divide this denominator by the denominator of the first fraction, and multiply the terms of the fraction by the quotient.

Proceed in a similar manner with each of the other fractions.

All fractions should first be reduced to lowest terms.

2. $\frac{2}{3}, \frac{3}{4}$.

3. $\frac{5}{6}, \frac{7}{8}$.

4. $\dfrac{2bx}{ay}, \dfrac{3ay}{2x}$.

5. $\dfrac{ax^2}{ab^2c}, \dfrac{d^2x}{bc^2d}$.

6. $\dfrac{x-y}{z}, \dfrac{z}{x+y}$.

7. $\dfrac{bc}{a-1}, \dfrac{ab}{a+1}$.

8. $\dfrac{a-b}{3(a+b)}, \dfrac{2a}{(a+b)^2}$.

9. $\dfrac{2x-2y}{x^2-y^2}, \dfrac{x-y}{x+1}$.

10. $\dfrac{a+x}{a}, b, \dfrac{2}{c+d}$.

11. $\dfrac{a^2}{y+1}, \dfrac{2a}{y^2-1}, \dfrac{ab}{y-1}$.

12. $\dfrac{ab}{x^4-y^4}, \dfrac{bc}{x^2+y^2}, \dfrac{cd}{y^2-x^2}$.

13. $\dfrac{x^2}{x^4+x^2+1}, \dfrac{x}{x^3+1}, \dfrac{x^3}{x^3-1}$.

14. $\dfrac{y+3}{y^2-3y+2}, \dfrac{y-2}{y^2+2y-3}, \dfrac{y-1}{y^2+y-6}$.

ADDITION AND SUBTRACTION OF FRACTIONS

97. In algebra, subtraction of fractions practically reduces to addition of fractions, for every fraction to be subtracted is added with its sign changed.

EXERCISES

98. 1. Find the algebraic sum of $\frac{4ax}{x^2-a^2}-\frac{a+x}{a-x}-\frac{x-a}{x+a}$.

SOLUTION.
$$\begin{aligned}\frac{4ax}{x^2-a^2}-\frac{a+x}{a-x}-\frac{x-a}{x+a}&=\frac{4ax}{x^2-a^2}+\frac{x+a}{x-a}-\frac{x-a}{x+a}\\&=\frac{4ax+(x+a)^2-(x-a)^2}{x^2-a^2}\\&=\frac{4ax+x^2+2ax+a^2-x^2+2ax-a^2}{x^2-a^2}\\&=\frac{8ax}{x^2-a^2}.\end{aligned}$$

RULE. — *Reduce the fractions to respectively equivalent fractions having their lowest common denominator.*

Change the signs of all the terms of the numerators of fractions preceded by the sign —, then find the sum of the numerators, and write it over the common denominator.

Reduce the resulting fraction to its lowest terms, if necessary.

Add:

2. $\frac{3a}{4}$ and $\frac{5a}{6}$.
3. $\frac{3b}{4c}$ and $\frac{-b}{3c}$.
4. $\frac{-x}{3y}$ and $\frac{-2x}{5y}$.

Subtract:

5. $\frac{3y}{8}$ from $\frac{5y}{6}$.
6. $\frac{-2a}{x}$ from $\frac{3x}{a}$.
7. $\frac{c-d}{2}$ from $\frac{c+d}{3}$.

Reduce these mixed expressions to fractions:

8. $x+\frac{1}{x}$.
9. $y^2-\frac{y^2}{3}$.
10. $\frac{a}{c}+b$.
11. $a^2+ab+\frac{2}{b}$.
12. $x+\frac{ax+c}{x^2}$.
13. $\frac{a-1}{2}+5a$.
14. $x+\frac{x^2-xy}{y}$.
15. $r-\frac{r-s+t}{3}$.
16. $x^2-x-\frac{x^2y}{x+y}$.

Simplify:

17. $\dfrac{b-c}{bc}-\dfrac{a-c}{ac}$.

18. $\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}$.

19. $\dfrac{3a}{5}+\dfrac{b}{2}-3+\dfrac{1}{b}$.

20. $\dfrac{a^2}{ab-b^2}+\dfrac{b^2}{ab-a^2}$.

21. $2a-3b-\dfrac{4a^2+9b^2}{2a+3b}$.

22. $\dfrac{a+1}{a^2+a+1}+\dfrac{a-1}{a^2-a+1}$.

23. $\dfrac{x+33}{x^2-9}-\dfrac{6}{x-3}+\dfrac{10}{x+3}$.

24. $\dfrac{a-b}{2(a+b)}+\dfrac{a^2+b^2}{a^2-b^2}+\dfrac{a}{b-a}$.

25. $\dfrac{a}{a-b}+\dfrac{b}{a+b}+\dfrac{a^2+b^2}{b^2-a^2}$.

26. $\dfrac{1}{a^3+8}-\dfrac{1}{8-a^3}+\dfrac{1}{4-a^2}$.

27. $\dfrac{5(x-3)}{x^2-x-2}-\dfrac{2(x+2)}{x^2+4x+3}-\dfrac{x-1}{6-x-x^2}$.

28. $\dfrac{x^2+x+1}{x^2-x+1}-1+\dfrac{2x}{x^2+x+1}$.

Suggestion. — Reduce the first fraction to a mixed number.

29. $\dfrac{a^2+2ab+b^2}{a^2+b^2}-1+\dfrac{2ab}{a^2-b^2}$.

30. $\dfrac{x+1}{x-1}+\dfrac{x-1}{x+1}-\dfrac{x+2}{x-2}-\dfrac{x-2}{x+2}$.

31. $\dfrac{a+x}{a-x}+\dfrac{a^2+x^2}{a^2-x^2}-\dfrac{a-x}{a+x}-\dfrac{a^2-x^2}{a^2+x^2}-\dfrac{4a^3x+4ax^3}{a^4-x^4}$.

Suggestion. — Combine the first two fractions, then the result and the third fraction, then this result and the fourth fraction, and so on.

32. $\dfrac{c^2ab}{(c-a)(b-c)}-\dfrac{b^2ca}{(b-a)(b-c)}-\dfrac{a^2bc}{(a-b)(a-c)}$.

Suggestion. — Change the signs of the factors $(c-a)$ and $(b-a)$.

33. $\dfrac{c+a}{(a-b)(b-c)}-\dfrac{b+c}{(c-a)(b-a)}+\dfrac{a+b}{(c-b)(a-c)}$.

MULTIPLICATION OF FRACTIONS

99. As in arithmetic,

Principle. — *The product of two or more fractions is equal to the product of their numerators divided by the product of their denominators.*

EXERCISES

100. 1. Simplify $\dfrac{a^4-b^4}{2a+2b} \times 4a \times \dfrac{a-b}{a^2+b^2} \times \dfrac{1}{2ab}$.

Solution. $\dfrac{a^4-b^4}{2a+2b} \times 4a \times \dfrac{a-b}{a^2+b^2} \times \dfrac{1}{2ab}$

$$= \frac{\cancel{(a^2+b^2)}\cancel{(a+b)}(a-b)}{\cancel{2}\cancel{(a+b)}} \times \frac{\cancel{2}\cdot\cancel{2}\cancel{a}}{1} \times \frac{a-b}{\cancel{a^2+b^2}} \times \frac{1}{\cancel{2}\cancel{a}b} = \frac{(a-b)^2}{b}.$$

Rule. — *Reduce integers and mixed numbers to fractions.*

Factor each numerator and each denominator.

Cancel factors common to numerator and denominator.

Write the product of the remaining factors in the numerator over the product of the remaining factors in the denominator.

Simplify:

2. $ab \times \dfrac{b}{a}$.

3. $2z \times \dfrac{x}{z^2}$.

4. $xy \times \dfrac{4}{y^3}$.

5. $\dfrac{3x}{2a^2} \times a^2b$.

6. $\dfrac{cd^3}{a^2x^2} \times \dfrac{a}{c^2d^2}$.

7. $\dfrac{8cd^2}{3ab} \times \dfrac{6a^2b}{8d^3}$.

8. $\dfrac{a^3y}{b^2x} \times \dfrac{-b^3}{a^2y}$.

9. $\dfrac{-xz}{ad} \times \dfrac{-d}{z^2}$.

10. $\dfrac{a^{m+1}}{b^{m+2}} \times \dfrac{b^{m+1}}{a^m}$.

11. $\dfrac{x}{y} \times \dfrac{y}{z} \times \dfrac{z}{x}$.

12. $\dfrac{a}{bc} \times \dfrac{b}{ac} \times \dfrac{c}{ab}$.

13. $\dfrac{x^2}{yz} \times \dfrac{y^2}{xz} \times \dfrac{z^2}{a^2}$.

14. $\dfrac{x^2y}{ab^2} \times \dfrac{a^2c}{y^2z} \times \dfrac{bz}{c^2x}$.

15. $\dfrac{a^2m}{b^2c} \times \dfrac{x^3}{a^3} \times \dfrac{ac^2}{mx^2}$.

16. $8a^2 \times \dfrac{b}{a+b} \times \dfrac{a^2-b^2}{4ab^2}$.

17. $\dfrac{5x^2y}{6ab} \times \dfrac{a-b}{xy} \times \dfrac{a^2b^2}{b-a}$.

18. $\dfrac{ac+ad}{c^2-d^2} \times 4c^2d \times \dfrac{3c-3d}{2acd^2}$.

19. $\dfrac{p^2-q^2}{p^2+q^2} \cdot \dfrac{r^2}{(p+q)^2} \cdot \dfrac{p^4-q^4}{s(p-q)^2}$.

20. $\dfrac{(x-y)^2}{x+y} \cdot \dfrac{x}{xy-y^2} \cdot \dfrac{(x+y)^2}{x^2-y^2}$.

Simplify:

21. $\dfrac{a^m b^n}{4x} \times \dfrac{6x^2}{a^{m+1}b^{2n}}$.

22. $\dfrac{x-4}{x+2} \times \dfrac{4-x^2}{16-x^2}$.

23. $\dfrac{a^2+ab}{a^2-b^2} \times \dfrac{a^3-b^3}{ab(a+b)}$.

24. $\dfrac{x^2+2x}{x^2-3x} \times \dfrac{x^2-x-6}{x^2+4x+4}$.

25. $\dfrac{x^2+3x+2}{x+4} \cdot \dfrac{x+2}{x^2+5x+4}$.

26. $\dfrac{a^2}{x^2+xy+y^2} \cdot \dfrac{x^4+x^2y^2+y^4}{ax^3+ay^3}$.

27. $\dfrac{a^2b+ab^2+b^3}{a^2-ab} \cdot \dfrac{(a-b)^2}{a^3-b^3}$.

28. $\dfrac{x^2+3x+2}{x^2-3x-10} \cdot \dfrac{x^2-6x+5}{x^2+8x+7}$.

29. $\dfrac{a^2+ab+2a+2b}{ax-2ay+2x-4y} \cdot \dfrac{x^2-2xy}{(a+b)^2}$.

30. $\dfrac{a^4-b^4}{a^3+b^3} \cdot \dfrac{a+b}{a^3-ab^2} \cdot \dfrac{a^2-ab+b^2}{(a+b)^2}$.

31. $\dfrac{x+y+z}{x+y-z} \cdot \dfrac{x-y+z}{x-y-z} \cdot \dfrac{(x-y)^2-z^2}{(x+y)^2-z^2}$.

32. $\dfrac{x^2+6x+8}{x^2+x-2} \cdot \dfrac{(x-1)^2}{x^2-4} \cdot \dfrac{x^2+5x+6}{x^2+3x-4}$.

33. $\dfrac{c^3+d^3}{c^3-d^3} \cdot \dfrac{(c^2-d^2)(c-3)}{(c-1)(c+d)^2} \cdot \dfrac{c^2+c-2}{c^2-c-6}$.

34. $\dfrac{a^3+b^3}{a(a^2+b^2)} \cdot \dfrac{a^4+2a^2b^2+b^4}{b^2(a^2-ab+b^2)} \cdot \dfrac{2ab}{(a+b)^2}$.

35. $\dfrac{x^2+2x+2}{x^2-x+1} \cdot \dfrac{x^3-1}{x^4+4} \cdot \dfrac{x^3+1}{x^2+x+1}$.

36. $\left(\dfrac{2}{3x}-1\right) \cdot \dfrac{9x^2y}{4-9x^2} \cdot \dfrac{2+3x}{4y^3}$.

37. $\left(a+\dfrac{ab}{a-b}\right)\left(a-\dfrac{ab}{a+b}\right)\left(\dfrac{a^2-b^2}{a^2+b^2}\right)$.

38. $\left(1-\dfrac{y+1}{y^2+7y+10}\right)\left(1-\dfrac{y+7}{y^2+7y+12}\right)$.

39. $\left(\dfrac{a^2+3ab+b^2}{ab}-1\right)\left(1-\dfrac{a^2-2ab-b^2}{a^2-b^2}\right)$.

DIVISION OF FRACTIONS

101. What is the **reciprocal** of a number? of a fraction? Write the reciprocal of 2; of $\frac{3}{4}$; of a; of $\frac{x}{y}$; of $\frac{1}{n}$.

102. As in arithmetic,

PRINCIPLE. — *Dividing by a fraction is equivalent to multiplying by its reciprocal.*

EXERCISES

103. 1. Divide $\dfrac{a^3+b^3}{a^2-b^2}$ by $\dfrac{a^2+ab+b^2}{a-b}$.

SOLUTION. $\dfrac{a^3+b^3}{a^2-b^2} \div \dfrac{a^2+ab+b^2}{a-b} = \dfrac{a^3+b^3}{a^2-b^2} \times \dfrac{a-b}{a^2+ab+b^2}$

$$= \frac{\cancel{(a+b)}(a^2-ab+b^2)}{\cancel{(a+b)}\cancel{(a-b)}} \times \frac{\cancel{a-b}}{a^2+ab+b^2} = \frac{a^2-ab+b^2}{a^2+ab+b^2}.$$

RULE. — *Reduce integers and mixed numbers to fractions. Take the reciprocal of each divisor and proceed as in multiplication.*

Simplify:

2. $1 \div \dfrac{c}{d}$.

3. $1 \div \dfrac{a^2}{b^2}$.

4. $\dfrac{2}{a} \div \dfrac{4}{a^2}$.

5. $\dfrac{2\,ab}{3\,xy} \div \dfrac{4\,a^2b}{9\,x^2y^2}$.

6. $\dfrac{12\,a^4b}{25\,ac} \div \dfrac{4\,ax}{15\,c^2}$.

7. $\dfrac{4\,xyz}{5} \div 2\,xy$.

8. $\dfrac{6\,m^2n^2}{5\,ax} \div \dfrac{12\,mn^2}{15\,a^3}$.

9. $\dfrac{a+b}{4\,a} \div \dfrac{a^2-b^2}{2\,b^2}$.

10. $\dfrac{(x+y)^2}{x^2-y^2} \div \dfrac{ax+ay}{x-y}$.

11. $(8\,y+4) \div \dfrac{2\,y+1}{3\,x}$.

12. $\dfrac{rs-s^2}{(r+s)^2} \div \dfrac{s^2}{r^2-s^2}$.

13. $\dfrac{x^3+a^3}{a^2-x^2} \div \dfrac{a^2+ax+x^2}{a-x}$.

14. $\dfrac{x^3-xz^2}{(x+z)^2} \div \dfrac{(x-z)^2}{x^2z-z^3}$.

15. $\left(a \div \dfrac{1}{b}\right) \div \left(b^2 \div \dfrac{1}{a^2}\right)$.

16. $\dfrac{a^2+3\,a-4}{a^2-1} \div \dfrac{a^2-16}{a^2+a}$.

17. $\dfrac{x^4-y^4}{x^2+2\,xy+y^2} \div \dfrac{x^2+y^2}{x^2+xy}$.

18. $\dfrac{m^4x+m^5}{m^3x-mx^3} \div \dfrac{m^3x^2-mx^4}{m^3x^3-x^6}$.

Simplify:

19. $\dfrac{a^3+27}{a^3-27} \div \dfrac{a+3}{a^2+3a+9}$.

20. $\dfrac{y^2+6y-7}{y^2+3y-4} \div \dfrac{y^2+4y-21}{2y+8}$.

21. $\dfrac{2a^2+a-15}{3a^2-a-2} \div \dfrac{2a^2-3a-5}{3a^2-7a-6}$.

22. $\dfrac{a^2+b^2-c^2+2ab}{a^2-b^2-c^2+2bc} \div \dfrac{a+b+c}{a-b+c}$.

23. $\left(r^2+\dfrac{1}{r^2}+2\right) \div \left(r+\dfrac{1}{r}\right)$.

24. $\left(1+\dfrac{1}{x^2}+\dfrac{1}{x^4}\right) \div \left(1+\dfrac{1}{x}+\dfrac{1}{x^2}\right)$.

25. $\left(y-3+\dfrac{-5}{y+1}\right) \div \left(2-\dfrac{7y+2}{y^2-1}\right)$.

26. $\left(x+1+\dfrac{1}{x}+\dfrac{1}{x^2}\right) \div \left(x+1-\dfrac{1}{x}-\dfrac{1}{x^2}\right)$.

27. $(a^3+b^3) \div \left(\dfrac{a^4+a^2b^2+b^4}{4x} \div \dfrac{a^2+ab+b^2}{ax-bx}\right)$.

Complex Fractions

104. Since a **complex fraction** is only an expression of unexecuted division, it may be simplified by performing the division.

EXERCISES

105. Simplify:

1. $\dfrac{\dfrac{a+b}{2}}{\dfrac{a^2-b^2}{6a}}$.

2. $\dfrac{x-\dfrac{1}{x}}{1+\dfrac{1}{x}}$.

3. $\dfrac{1+\dfrac{3}{x}}{\dfrac{x^2-9}{2x^2}}$.

4. $\dfrac{m-\dfrac{3m}{x}}{x-\dfrac{x}{m}}$.

5. $\dfrac{\dfrac{15}{a}-2+a}{1-\dfrac{5}{a}}$.

6. $\dfrac{\dfrac{x+y}{y}-\dfrac{x+y}{x}}{\dfrac{1}{y}-\dfrac{1}{x}}$.

Simplify:

7. $$\frac{\frac{1}{x}+\frac{4}{x^2}+\frac{4}{x^3}}{1+\frac{5}{x}+\frac{6}{x^2}}.$$

8. $$\frac{\frac{x-5}{2}-7+\frac{24}{x}}{\frac{9-3x}{x}}.$$

9. $$\frac{\frac{1}{x+y}+\frac{2}{x-y}-\frac{9}{3x-y}}{\frac{-8y}{y^2-9x^2}}.$$

10. $$\frac{\frac{x^2+(a+b)x+ab}{x^2-(a+b)x+ab}}{\frac{x^2-b^2}{x^2-a^2}}.$$

11. $$\frac{3xyz}{yz+xz+xy}-\frac{\frac{x-1}{x}+\frac{y-1}{y}+\frac{z-1}{z}}{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}}.$$

A complex fraction of the form $\dfrac{a}{b+\dfrac{c}{d+\cdots}}$ is called a **continued fraction.**

Every continued fraction may be simplified by successively simplifying its *last complex part* by multiplying both terms by the last denominator.

12. $$\frac{1}{1+\frac{1}{1+\frac{1}{a}}}.$$

13. $$\frac{1}{c-1+\frac{1}{1+\frac{c}{4-c}}}.$$

14. $$\frac{1+a^3}{1-\frac{a}{1+\frac{a}{1-a}}}.$$

15. $$\frac{y}{y+\frac{y}{y+\frac{1}{y-1}}}.$$

16. $$\frac{x}{x+1-\frac{x}{x+\frac{x}{x-1}}}.$$

17. $$\frac{a}{a+1+\frac{a}{a+1-\frac{1}{a}}}.$$

18. $$1+\frac{c}{1+c+\frac{2c}{1+\frac{1}{c}}}.$$

19. $$a+\frac{1}{a-1-\frac{2a}{2+\frac{1}{a}}}.$$

MISCELLANEOUS EXERCISES

106. Reduce to lowest terms:

1. $\dfrac{5x^2-11x-12}{10x^2+23x+12}$.

2. $\dfrac{x^3+3x^2-x-3}{x^3+3x^2+x+3}$.

3. $\dfrac{a^3+b^3}{a^4+a^2b^2+b^4}$.

4. $\dfrac{cx-cd}{cx+3x-3d-cd}$.

Simplify:

5. $\dfrac{x}{2y-1}+\dfrac{y}{2y+1}-\dfrac{y-x}{1-4y^2}$.

6. $\left(1+\dfrac{x}{a-x}\right)\left(\dfrac{x}{x+a}-\dfrac{2x^2+2ax-a^2}{x^2+3ax+2a^2}\right)$.

7. $\left(\dfrac{m-3n}{m+n}\right)\left(1+\dfrac{4n}{m+n}\right)\div\left(\dfrac{m}{n}+2-\dfrac{15n}{m}\right)$.

8. $\left(x^2-3xy-2y^2+\dfrac{12y^3}{x+3y}\right)\div\left(3x-6y-\dfrac{2x^2}{x+3y}\right)$.

9. $\dfrac{\dfrac{c}{(a+1)^2}+\dfrac{d}{(a+1)^2}}{\dfrac{a}{(a+1)^4}+\dfrac{1}{(a+1)^4}}$.

10. $\dfrac{1}{1-\dfrac{1}{1-\dfrac{1}{1-x}}}$.

11. $\dfrac{\dfrac{m^3-n^3}{m^3+n^3}\left(1-\dfrac{2n}{m+n}\right)}{1+\dfrac{2mn}{m^2-mn+n^2}}$.

12. $\dfrac{1}{2-\dfrac{3}{4-\dfrac{5}{6-x}}}$.

13. $\dfrac{\left(\dfrac{a}{x^2}+\dfrac{1}{x}+\dfrac{1}{a}+\dfrac{x}{a^2}\right)\left(\dfrac{a^2}{x^3}-\dfrac{1}{x}+\dfrac{x}{a^2}\right)}{\dfrac{a^3}{x^5}\left(1+\dfrac{x}{a}\right)}$.

14. $\dfrac{x^3-\dfrac{8}{y^3}}{x^3y^3-x^2y^2}\times\dfrac{\dfrac{1}{xy}+\dfrac{1}{x^2y^2}}{1+\dfrac{2}{xy}+\dfrac{4}{x^2y^2}}\times\dfrac{xy-1}{xy+1}$.

SIMPLE EQUATIONS

ONE UNKNOWN NUMBER

107. Review the definition, explain, and illustrate:

1. **Numerical equation.**
2. **Literal equation.**
3. **Integral equation.**
4. **Fractional equation.**
5. **Identical equation (identity).**
6. **Equation of condition.**
7. **Root of an equation.**
8. **Solution of an equation.**
9. **Equivalent equations.**
10. **Simple equation.**

11. Give two other names that are applied to simple equations.

12. When is an equation said to be **satisfied**?

108. By the axioms in § 43, if the members of an equation are increased or diminished or multiplied or divided by the same or equal numbers, the two resulting members are *equal* and form an equation. But it does not necessarily follow that the equation so formed is *equivalent* to the given equation.

For example, if both members of the equation $x + 2 = 5$, whose only root is $x = 3$, are multiplied by $x - 1$, the resulting numbers, $(x + 2)(x - 1)$ and $5(x - 1)$, are *equal* and form an equation,

$$(x + 2)(x - 1) = 5(x - 1),$$

which is not *equivalent* to the given equation, since it is satisfied by $x = 1$ as well as by $x = 3$; that is, the root $x = 1$ has been *introduced.*

In applying axioms to the solution of equations we endeavor to change to *equivalent* equations, each simpler than the preceding, until an equation is obtained having the unknown number in one member and the known numbers in the other.

109. The following principles serve to guard against *introducing* or *removing* roots without accounting for them:

Principles. — 1. *If the same expression is added to or subtracted from both members of an equation, the resulting equation is equivalent to the given equation.*

2. *If both members of an equation are multiplied or divided by the same known number, except zero, the resulting equation is equivalent to the given equation.*

3. *If both members of an integral equation are multiplied by the same unknown integral expression, the resulting equation has all the roots of the given equation and also the roots of the equation formed by placing the multiplier equal to zero.*

It follows from Principle 3 that it is *not allowable to remove from both members of an equation a factor that involves the unknown number, unless the factor is placed equal to zero and the root of this equation is preserved.* Thus, if $x-2$ is removed from both members of the equation $(x-2)(x+4)=7(x-2)$, the resulting equation $x+4=7$ has only the root $x=3$; consequently, the root of $x-2=0$, *removed* by dividing by the factor $x-2$, should be preserved.

Clearing Equations of Fractions

EXERCISES

110. 1. Solve $\dfrac{3x-5}{4}-\dfrac{7x-13}{6}=3-\dfrac{x+3}{2}$.

Solution. — Multiply both members of the equation by the l. c. d., which in this case is 12, to **clear the equation of fractions**, obtaining,

$$3(3x-5)-2(7x-13)=36-6(x+3).$$

Expand, $$9x-15-14x+26=36-6x-18.$$

Transpose, etc., $$x=7.$$

Verification. — When $x=7$, the given equation becomes $-2=-2$, an identity; consequently, the equation is satisfied for $x=7$.

Rule. — *To clear an equation of fractions, multiply both members by the lowest common denominator of the fractions.*

1. Reduce all fractions to lowest terms and unite fractions that have a common denominator before clearing.
2. Discover **extraneous** roots by verification, and reject them.

Solve, and verify each result:

2. $x+\frac{x}{4}=5.$

3. $\frac{x}{3}-10=\frac{25}{3}.$

4. $\frac{x}{6}+2x=26.$

5. $\frac{x}{5}+\frac{x}{7}=24.$

6. $\frac{3x}{4}+\frac{7x}{16}-\frac{x}{2}-\frac{5x}{16}=\frac{1}{8}.$

7. $\frac{2x}{15}+\frac{5x}{25}-\frac{4x}{9}+\frac{x}{6}=\frac{1}{9}.$

8. $\frac{3x}{4}-\frac{7x}{12}=\frac{11x}{36}-\frac{8x}{9}+\frac{3}{2}.$

9. $\frac{15x}{7}+\frac{5x}{6}-\frac{11x}{3}+\frac{19x}{14}=2.$

10. $\frac{v+1}{3}-\frac{v+4}{5}+\frac{v+3}{4}=16.$

11. $\frac{5x-6}{5}+\frac{4x+7}{10}=\frac{1}{2}+\frac{3x-4}{5}.$

12. $\frac{x+1}{6}-\frac{x-2}{5}+\frac{x+3}{10}=4.$

13. $\frac{3y+4}{4}+\frac{y-3}{3}-\frac{4-2y}{6}=5.$

14. $\frac{3x-5}{2}-\frac{x+1}{4}=\frac{2x}{7}+\frac{5x-11}{6}.$

15. $\frac{10x+3}{3}-\frac{6x-7}{2}=10(x-1).$

16. $.7x+.24=.08x+9.2-.02x.$

SUGGESTION. — Clear of decimal fractions by multiplying by 100.

17. $.375-.25x+.625=.5x-.6+.05x.$

18. $.18x-28.4-.06x=.35-.2x-9.55.$

19. $\frac{n+4}{.3}+\frac{2-2n}{.6}=\frac{n+1}{.2}-\frac{10}{.3}.$

20. $\frac{2x}{x+3}+\frac{x}{x-5}-3=\frac{1}{2x-10}.$

21. $\dfrac{6r-7}{9r+6}-\dfrac{5(r+1)}{12r+8}=\dfrac{1}{12}.$

22. $\dfrac{10q+17}{18}-\dfrac{5q-2}{9}=\dfrac{12q-9}{11q-8}.$

23. $\dfrac{y-1}{y-2}-\dfrac{y-2}{y-3}=\dfrac{y-4}{y-5}-\dfrac{y-5}{y-6}.$

SUGGESTION. — Combine the fractions in each member of the equation before clearing of fractions.

24. $\dfrac{2x+1}{x+1}-\dfrac{2x+9}{x+5}=\dfrac{x-3}{x-4}-\dfrac{x-7}{x-8}.$

25. $\dfrac{x^3+2}{x+1}-\dfrac{x^3-2}{x-1}=\dfrac{10}{x^2-1}-2x.$

26. $3.1416x-17.1441+.0216x=.2535.$

27. $\dfrac{3n-4}{4}-\left(\dfrac{4n}{5}+\dfrac{n+2}{2}\right)=\dfrac{9n}{10}+\left(19-\dfrac{n+4}{2}\right).$

28. $\dfrac{(x-3)^2}{7}-\dfrac{(x+4)^2}{3}=20-\left(8x+\dfrac{5x+10}{21}\right)-\dfrac{4x^2}{21}.$

29. $\dfrac{2x\left(1-\dfrac{5}{x}\right)}{3}+\dfrac{3x\left(1-\dfrac{4}{x}\right)}{4}=\dfrac{x-4}{\frac{4}{5}}.$

30. $\dfrac{1}{2}x-2\left(\dfrac{4x}{5}-3\right)=4-\dfrac{3}{2}\left(\dfrac{x}{2}+1\right).$

31. $\dfrac{(2x+1)^2}{.05}-\dfrac{(4x-1)^2}{.2}=\dfrac{15}{.08}+\dfrac{3(4x+1)}{.4}.$

32. $\dfrac{17+\dfrac{3}{x}}{3}+\dfrac{1+\dfrac{18}{x}}{5}=\dfrac{\dfrac{21}{x}-1}{9}+\dfrac{\dfrac{100}{x}+\dfrac{5}{3}}{15}.$

33. $\dfrac{\frac{1}{4}(x-4)}{\frac{3}{2}}-\dfrac{4x-16}{6}=\dfrac{3}{5}-\dfrac{\dfrac{2x}{5}+5}{\frac{5}{2}}.$

Literal Equations

111. 1. Solve the equation $\dfrac{x+m^2}{n}=\dfrac{x+n^2}{m}$ for x.

SOLUTION. — Clear the equation of fractions, obtaining

$$mx+m^3=nx+n^3.$$

Transpose, etc., $\quad mx-nx=n^3-m^3,$

or $\quad (m-n)x=-(m^3-n^3).$

Divide by $m-n$, $\quad x=-(m^2+mn+n^2).$

VERIFICATION. — Let $m=2$ and $n=1$; then, $x=-(4+2+1)=-7$, and the given equation becomes $-3=-3$, an identity; that is, $-(m^2+mn+n^2)$ is the root.

Solve for x, and verify each result:

2. $b^3+ax=a^3+bx.$

3. $x(1-3c)+9c^2=1.$

4. $\dfrac{a}{bx}+\dfrac{b}{ax}=a^2+b^2.$

5. $am-b+\dfrac{x}{m}=\dfrac{ax}{b}.$

6. $\dfrac{x-2c}{ax-4a^2}=\dfrac{2}{c}.$

7. $\dfrac{x+r}{x-s}=\dfrac{r+s}{r-s}.$

8. $x-1+4b=b(3b+x).$

9. $(x-a)^2-(x-b)^2=(a-b)^2.$

10. $\dfrac{a(x-a)}{b}+\dfrac{b(x-b)}{a}=x.$

11. $\dfrac{b}{a+b}=\dfrac{(a+b)^2-a(a+b)}{x}.$

12. $\dfrac{3ax-2b}{3b}-\dfrac{ax-a}{2b}=\dfrac{ax}{b}-\dfrac{2}{3}.$

13. $\dfrac{x-2ab}{cx}-\dfrac{1}{x}=\dfrac{x-3c}{abx}.$

14. $x(b+c)-2a(b+c)=a^2-ax+b^2+c(2b+c).$

15. $a(x-a-b)+b(x-a+2c)=c(x-2a+c)+b^2.$

16. $(x+c)(x-d)-2(x+d)(x-c)=c^2-(x-d)(x-c).$

17. $\dfrac{1}{a(b-x)}+\dfrac{1}{b(c-x)}-\dfrac{1}{a(c-x)}=0.$

18. $\dfrac{x+a}{b}+\dfrac{x+c}{a}+\dfrac{x+b}{c}=\dfrac{a}{b}+\dfrac{b}{c}+\dfrac{c}{a}+1.$

19. $\dfrac{x}{a+b+c}+\dfrac{x}{a+b-c}=a^2+b^2+c^2+2ab.$

20. $\dfrac{x^2-ax-bx+ab}{x-a}=\dfrac{x^2-2bx+2b^2}{x-b}-\dfrac{c^2}{x-c}.$

Problems

112. Reread the general directions given in § 47, and solve:

1. Leo has 3 times as many plums as Carl. If each had 5 more, Leo would have only twice as many as Carl. How many plums has each?

2. Ann paid \$ 3.00 for three books. The first cost $\frac{1}{2}$ as much as the second and $\frac{1}{3}$ as much as the third. Find the cost of each.

3. Cornstalk paper costs $\frac{1}{3}$ as much as paper made from rags. A ton of the former costs \$ 50 less than one of the latter. Find the cost of each kind of paper per ton.

4. Four wagons drew 38 logs from the woods, one wagon holding 2 logs more than each of the others. How many logs did each wagon hold?

5. The distance around a desk top is 170 inches. If the desk top is 15 inches longer than it is wide, how wide is it?

6. A man paid \$ 300 for a horse, a harness, and a carriage. The carriage cost twice as much as the harness, and the horse as much as the harness and carriage together. Find the cost of each.

7. I bought 15 books for \$ 6.60, spending 30 cents each for one kind and 60 cents each for the other. How many books of each kind did I buy?

8. A shipment of 12,000 tons of coal arrived at Boston on 3 barges and 2 schooners. Each schooner held $3\frac{1}{2}$ times as much as each barge. Find the capacity of a barge; of a schooner.

9. John has \$ 6.75. He has 3 times as many dimes as nickels, and as many quarters as the sum of the nickels and dimes. How many coins has he of each denomination?

10. John is 15 years older than Frank. In 5 years Frank's age will be $\frac{1}{2}$ John's age. What is the age of each?

11. George is $\frac{1}{2}$ as old as his father; a years ago he was $\frac{1}{3}$ as old as his father. What is the age of each?

12. Harold is n times as old as his brother; r years ago he was m times as old. Find the age of each.

13. Three pails and 6 baskets contain 576 eggs. All the pails contain $\frac{1}{3}$ as many eggs as all the baskets. How many eggs are there in each pail? in each basket?

14. The cost per mile of running a train was 14 cents less with electrical equipment than with steam, or $\frac{3}{5}$ as much. What was the cost per mile with electricity?

15. A rectangle is 9 feet longer than it is wide. A square whose side is 3 feet longer than the width of the rectangle is equal to the rectangle in area. What are the dimensions of the rectangle?

16. A field is twice as long as it is wide. By increasing its length 20 rods and its width 30 rods, the area will be increased 2200 square rods. What are its dimensions?

17. The length of the steamship *Mauretania* is 790 feet, or 2 feet less than 9 times its width. What is its width?

18. The length of a tunnel was $22\frac{1}{2}$ times its width. If the length had been 50 feet less, it would have been 20 times the width. Find its length; its width.

19. In a purse containing \$1.45 there are $\frac{1}{2}$ as many quarters as 5-cent pieces and $\frac{2}{3}$ as many dimes as 5-cent pieces. How many coins are there of each kind?

20. The St. Lawrence River at a point where it is spanned by a bridge is 1800 feet wide. This is 180 feet less than $\frac{3}{5}$ of the length of the bridge. How long is the bridge?

21. A girl found that she could buy 18 apples with her money and have 5 cents left, or 12 oranges and have 11 cents left, or 8 apples and 6 oranges and have 10 cents left. How much money had she?

22. A can do a piece of work in 10 days. If B can do it in 12 days, in how many days can both do it?

SUGGESTION. — Let $x =$ the required number of days.

Then, $\frac{1}{x} =$ the part of the work both can do in 1 day.

23. A can pave a walk in 6 days, and B in 8 days. How long will it take A to finish the job after both have worked 3 days?

24. A can do a piece of work in $2\frac{1}{2}$ days and B in $3\frac{1}{3}$ days. In how many days can both do it?

25. A can paint a barn in 12 days, and B and C in 4 days. In how many days can all together do it?

26. A and B can lay a walk in 8 days, B and C in 9 days, and A and C in 12 days. In how many days can C do the work alone?

27. One pipe can fill a tank in 45 minutes and another can fill it in 55 minutes. How long will it take both to fill it?

28. A tank can be filled by one pipe in a hours, by a second pipe in c hours, and emptied by a third in b hours. If all are open, how long will it take to fill the tank?

29. In a number of two digits, the tens' digit is 3 more than the units' digit. If the number less 6 is divided by the sum of its digits, the quotient is 6. Find the number.

SUGGESTION. — Let $x =$ the digit in units' place.

Then, $x + 3 =$ the digit in tens' place,

and $10(x + 3) + x =$ the number.

30. The sum of the digits of a two-digit number is 11. 63 added to the number reverses the digits. Find the number.

31. In a two-digit number, the tens' digit is 5 more than the units' digit. If the digits are reversed, the number thus formed is $\frac{3}{8}$ of the original number. Find the number.

32. In a two-digit number, the units' digit is 3 more than the tens' digit. If the number with digits reversed is multiplied by 8, the result is 14 times the original number. Find the number.

33. A man invests \$5650, part at 4 % and the remainder at 6 %. His annual income is \$298. How much has he invested at each rate?

34. A man has $\frac{2}{3}$ of his property invested at 4 %, $\frac{1}{4}$ at 3 %, and the remainder at 2 %. How much is his property valued at, if his annual income is \$860?

35. Mr. Johnson had \$15,000 invested, part at 6 % and part at 3 %. If his annual return was 5 % of the total investment, what amount was invested at each rate?

36. A man desires to secure an income on \$12,000 which shall be at the rate of $4\frac{1}{2}$ %. He buys two kinds of bonds which yield 6 % and 4 %, respectively. How much does he invest in each?

37. A bank invests s dollars, part at 6 % and the remainder at 5 %. If the annual income is m dollars, how much is invested at each rate?

38. My annual income is m dollars. If $\frac{1}{n}$ of my property is invested at 5 % and the remainder at 6 %, what is my capital?

39. At what time between 6 and 7 o'clock are the hands of a clock together?

SUGGESTION. — Let $x =$ the number of minute spaces passed over by the minute hand after 6 o'clock until the hands come together.

Then, $\frac{x}{12} =$ the number of minute spaces passed over by the hour hand.

Since the hands are 30 minute spaces apart at 6 o'clock, $x - \frac{x}{12} = 30$.

40. At what time between 2 and 3 o'clock are the hands of a clock at right angles to each other?

41. Find two different times between 6 and 7 o'clock when the hands of a clock are at right angles to each other.

42. Find at what time between 1 and 2 o'clock the minute hand of a clock forms a straight line with the hour hand.

43. I have $6\frac{1}{4}$ hours at my disposal. How far may I ride at the rate of 9 miles an hour, that I may return in the given time, walking back at the rate of $3\frac{1}{2}$ miles an hour?

SUGGESTION. — Let $x =$ the number of miles I may ride.

Then, the equation of the problem is $\frac{x}{9} + \frac{x}{3\frac{1}{2}} = 6\frac{1}{4}$.

44. A steamboat that goes 12 miles an hour in still water takes as long to go 16 miles upstream as 32 miles downstream. Find the velocity of the stream.

45. A motor boat went up the river and back in 2 hours and 56 minutes. Its rate per hour was $17\frac{1}{2}$ miles going up and 21 miles returning. How far up the river did it go?

46. A yacht sailed up the river and back in r hours. Its rate per hour was s miles going up and t miles returning. How far up the river did it sail?

47. A train moving 20 miles an hour starts 30 minutes ahead of another moving 50 miles an hour in the same direction. How long will it take the latter to overtake the former?

48. If an automobile had taken m minutes longer to go a mile, the time for a trip of d miles would have been t hours. How long did it take the automobile to go a mile?

49. In an alloy of 75 pounds of tin and copper there are 12 pounds of tin. How much copper must be added that the new alloy may be $12\frac{1}{2}$ % tin?

SUGGESTION. — Let $x =$ the number of pounds of copper to be added.

Since the new alloy weighs $(75 + x)$ pounds, the equation of the problem is $.12\frac{1}{2}(75 + x) = 12$.

50. In an alloy of 100 pounds of zinc and copper there are 75 pounds of copper. How much copper must be added that the alloy may be 10 % zinc?

51. In a solution of 60 pounds of salt and water there are 3 pounds of salt. How much water must be evaporated that the new solution may be 10 % salt?

52. In p pounds of bronze, the amount of tin was m times that of the zinc and n pounds less than $\frac{1}{r}$ that of the copper. How many pounds of zinc were there?

53. It is desired to add sufficient water to 6 gallons of alcohol 95 % pure to make a mixture 75 % pure. How many gallons of water are required?

54. How much pure gold added to 180 ounces of gold 14 carats fine ($\frac{14}{24}$ pure) will make an alloy 16 carats fine?

55. How much pure gold must be added to w ounces of gold 18 carats fine that the alloy may be 22 carats fine?

56. A body placed in a liquid loses as much weight as the weight of the liquid displaced. A piece of glass having a volume of 50 cubic centimeters weighed 94 grams in air and 51.6 grams in alcohol. How many grams did the alcohol weigh per cubic centimeter?

57. Brass is $8\frac{2}{5}$ times as heavy as water, and iron $7\frac{1}{2}$ times as heavy as water. A mixed mass weighs 57 pounds, and when immersed displaces 7 pounds of water. How many pounds of each metal does the mass contain?

Suggestion. — Let there be x pounds of brass and $(57 - x)$ pounds of iron. Then, x pounds of brass will displace $(x \div 8\frac{2}{5})$ pounds of water.

58. If 1 pound of lead loses $\frac{2}{23}$ of a pound, and 1 pound of iron loses $\frac{2}{15}$ of a pound when weighed in water, how many pounds of lead and of iron are there in a mass of lead and iron weighing 159 pounds in air and 143 pounds in water?

59. If tin and lead lose, respectively, $\frac{5}{37}$ and $\frac{2}{23}$ of their weights when weighed in water, and a 60-pound mass of lead and tin loses 7 pounds when weighed in water, what is the weight of the tin in this mass?

60. If 97 ounces of gold weigh 92 ounces when weighed in water, and 21 ounces of silver weigh 19 ounces when weighed in water, how many ounces of gold and of silver are there in a mass of gold and silver that weighs 320 ounces in air and 298 ounces in water?

Formulæ

113. A formula expresses a principle or a rule in symbols. The solution of problems in commercial life, and in mensuration, mechanics, heat, light, sound, electricity, etc., often depends upon the ability to solve and apply formulæ.

I. The lateral surface (S) of a circular cylinder is 2 times π ($= 3.1416$) times the product of the radius (r) of the base and the height (h), or
$$S = 2\pi rh.$$

1. Solve for h. Find the height of a circular cylinder whose lateral surface is 942.48 square inches and the radius of whose base is 10 inches.

II. The formula for the surface (S) of a rectangular solid in terms of its length (l), breadth (b), and height (h) is
$$S = 2(lb + lh + bh).$$

2. Solve for l; for b; for h.

3. Find the height of a rectangular solid 6 feet long and 4 feet wide, having a total surface of 108 square feet.

III. The formula for the percentage (p) in terms of the base (b) and rate (r) is
$$p = br.$$

4. Solve for b; for r.

5. If nickel-steel is 2.85 % nickel, how many pounds of nickel are there in 2 tons of the nickel-steel?

6. Out of a lot of 360 brass castings 24 were spoiled. What per cent of the castings were spoiled?

7. The machines in a shop require all together 16 horse power to run them and are driven by a single motor. If 20 % of the power of the motor is lost through friction, etc., what is the necessary horse power of the motor used?

IV. The formula for the interest (i) on a principal of p dollars at simple interest at r % for t years is
$$i = prt.$$

8. Solve for p; for t. What principal will yield \$480 interest in 3 years 4 months at 6 %?

9. In what time will \$4000 yield \$350 interest at 5 %?

V. The formula for the amount (a) of a sum of money (p) at the end of t years at simple interest at r % is

$$a = p(1 + rt).$$

10. Solve for p; for t. What principal will amount to \$ 828 in $3\frac{3}{4}$ years at 4 % ?

11. How long will it take \$ 600 to amount to \$ 1000 at 6 % ?

VI. The formula for converting a temperature of C degrees Centigrade into its equivalent temperature of F degrees Fahrenheit is

$$F = \frac{9}{5}C + 32.$$

12. Solve for C. Express 86° Fahr. in degrees Centigrade.

VII. If a steel rail at 0° C. is heated, for every degree it is heated it will expand a certain part of its original length. If E denotes the total expansion, L the original length, T the number of degrees change in temperature, and k the certain fractional multiplier, or coefficient of expansion; then

$$E = LkT.$$

13. Solve for k. A steel rail 30 feet long at 0° C. expanded to a length of 30.001632 feet at 50° C. Find the value of k.

VIII. The formula for the *velocity* acquired in t seconds by a body moving with uniform *acceleration* (a) is

$$V = at.$$

14. Solve the formula for a; for t.

15. A body starting from rest and moving with a uniform acceleration acquires a velocity of 100 feet per second in 5 seconds. What is the acceleration?

IX. The formula for the space (s) passed over by a freely falling body in any second (t) is

$$s = \frac{1}{2}g(2t - 1),$$

g, the acceleration due to gravity being approximately 32 feet.

16. Solve the formula for t. A brick dropped to the ground from the top of a chimney. How far did it fall during the second second? the third second?

X. The formula for the width (W) in inches of a nut for a bolt of a certain diameter (D) in inches is

$$W = \frac{3}{2}D + \frac{1}{8}.$$

17. Find the width of a nut for a $\frac{7}{8}$-inch bolt.

18. Solve the above formula for D. What is the diameter of a bolt that will fit a nut $1\frac{5}{8}$ inches wide?

XI. The length (l) of the belt required for two pulleys, each with a radius of r feet, equals the circumference of one pulley plus twice the distance (d) in feet between the centers of the pulleys, that is,

$$l = 2(\pi r + d).$$

19. Solve the formula for d; for r.

20. How far apart are the centers of two pulleys, radius $10\frac{1}{2}$ inches, if a belt $35\frac{1}{2}$ feet long is required? (Use $\pi = 3\frac{1}{7}$.)

XII. The length (L) of a bar of thickness (T) needed to make a welded ring with a certain inside diameter (D) is

$$L = \pi(D + T).$$

21. Find the length of a bar $\frac{1}{2}$ of an inch thick required to make a ring with an inside diameter of 10 inches. (Use $\pi = 3\frac{1}{7}$.)

22. Solve for D. Find the inside diameter of the ring that can be made from a bar 44 inches long and $\frac{1}{2}$ of an inch thick.

XIII. The cutting speed (S) of a tool is the rate in feet per minute at which the cutting tool passes over the surface being cut. It equals $\frac{1}{12}$ of the circumference (πd) of the piece being cut in inches multiplied by the number (n) of revolutions the cutting tool makes per minute, or

$$S = \frac{\pi dn}{12}.$$

23. The diameter of a piece of brass being turned in a lathe is $3\frac{1}{2}$ inches. If the lathe makes 120 revolutions per minute, what is the cutting speed? (Use $\pi = 3\frac{1}{7}$.)

24. Solve for n. The cutting speed of a lathe in turning a piece of iron $4\frac{2}{3}$ inches in diameter was 33 feet per minute. How many revolutions did the lathe make per minute?

RATIO AND PROPORTION

RATIO

114. Define and illustrate:

1. **Ratio; couplet.**
2. **Antecedent of a ratio.**
3. **Consequent of a ratio.**
4. **Duplicate ratio.**
5. **Triplicate ratio.**
6. **Reciprocal (inverse) ratio.**

115. The ratio of two quantities is the *ratio* of their *numerical measures*, when expressed *in terms of a common unit.*

Thus, the ratio of 33 ft. to 3 rd., or 2 rd. to 3 rd., is $\frac{2}{3}$.

116. One number is said to be **greater than** another when the remainder obtained by subtracting the second from the first is *positive*, and to be **less than** another when the remainder obtained by subtracting the second from the first is *negative.*

If $a - b$ is a positive number, a is *greater* than b; but if $a - b$ is a negative number, a is *less* than b.

Any negative number is regarded as less than 0; and, of two negative numbers, the one more remote from 0 is the less.

An algebraic expression indicating that one number is greater or less than another is called an **inequality.**

117. A ratio is said to be a ratio of **greater inequality**, a ratio of **equality**, or a ratio of **less inequality**, according as the antecedent is *greater than*, *equal to*, or *less than* the consequent.

Thus, when a and b are positive numbers, $\frac{a}{b}$ is a ratio of greater inequality, if $a > b$; a ratio of equality, if $a = b$; and a ratio of less inequality, if $a < b$.

Properties of Ratios

118. Since a ratio is expressed as a fraction, ratios have the same properties as fractions. Hence,

PRINCIPLES. — 1. *Multiplying or dividing both terms of a ratio by the same number does not change the value of the ratio.*

2. *Multiplying the antecedent or dividing the consequent of a ratio by any number multiplies the ratio by that number.*

3. *Dividing the antecedent or multiplying the consequent by any number divides the ratio by that number.*

4. *A ratio of greater inequality is decreased and a ratio of less inequality is increased by adding the same positive number to each of its terms.*

For, given the positive numbers a, b, and c, and the ratio $\frac{a}{b}$.

1. *When* $a > b$, *it is to be proved that* $\frac{a+c}{b+c} < \frac{a}{b}$.

$$\frac{a+c}{b+c} - \frac{a}{b} = \frac{c(b-a)}{b(b+c)}.$$

Since $a > b$, $b - a$ is negative, and $\frac{c(b-a)}{b(b+c)}$ is negative; therefore, $\frac{a+c}{b+c} - \frac{a}{b}$ is negative, and (§ 116) $\frac{a+c}{b+c} < \frac{a}{b}$.

2. *When* $a < b$, *it is to be proved that* $\frac{a+c}{b+c} > \frac{a}{b}$.

As in 1, since $a < b$, $\frac{a+c}{b+c} - \frac{a}{b}$ is positive, and $\frac{a+c}{b+c} > \frac{a}{b}$.

5. *In a series of equal ratios, the sum of all the antecedents is to the sum of all the consequents as any antecedent is to its consequent.*

For, given $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = r$, the value of each ratio.

By Ax. 3, $\quad a = br,\ c = dr,\ e = fr;$

whence, Ax. 1, $\quad a + c + e = (b + d + f)r;$

$$\therefore \frac{a+c+e}{b+d+f} = r \text{ or } \frac{a}{b} \text{ or } \frac{c}{d} \text{ or } \frac{e}{f}.$$

EXERCISES

119. 1. What is the ratio of $6a$ to $9a$? of $9a$ to $6a$?

2. What is the ratio of $\frac{1}{3}$ to $\frac{1}{2}$? $\frac{1}{2}x$ to $\frac{1}{4}x$? $\frac{3}{8}y$ to $\frac{3}{4}y$?

3. What is the inverse ratio of $5:8$? of $\frac{10}{9}$? of $\frac{7}{15}$?

4. Write the duplicate ratio of $2:3$; of $4:5$.

5. Write the triplicate ratio of $1:2$; of $3:4$.

Reduce to lowest terms the ratio expressed by:

6. $\frac{9}{15}$.	9. $\frac{6x}{9x}$.	12. $\frac{12abc}{30a^2bc}$.	15. $\frac{x+y}{3x+3y}$.
7. $\frac{12}{18}$.	10. $\frac{5a^2b}{10b^3}$.	13. $\frac{25x^2y^2}{40x^3y}$.	16. $\frac{a^2-b^2}{(a+b)^2}$.
8. $\frac{18}{24}$.	11. $\frac{8cx}{12x^2}$.	14. $\frac{18a^2b^3d}{48ab^2d^3}$.	17. $\frac{2x^2+2y^2}{x^4-y^4}$.

18. Two numbers are in the ratio of $4:5$. If 9 is subtracted from each, what is the ratio of the remainders?

19. When the ratio is $\frac{1}{2}$ and the consequent is $10ab$, what is the antecedent?

Find the value of each of the following ratios:

20. $1\frac{1}{2}:4\frac{1}{2}$.	23. $.5b:.6c$.	26. $(x^2-4):(x^3-8)$.
21. $\frac{1}{3}ax:\frac{2}{3}ay$.	24. $.4x^2:10x^3$.	27. $(a^6+b^6):(a^2+b^2)$.
22. $\frac{1}{2}bc^2:\frac{1}{4}b^2c$.	25. $\frac{5}{8}x^2y^2:\frac{1}{4}xy$.	28. $(a^3-1):(a^2+a+1)$.

29. Reduce the ratios $a:b$ and $x:y$ to ratios having the same consequent.

30. In an alloy of 78 ounces of silver and copper there are 6 ounces of silver. Find the ratio of silver to copper.

31. In a mixed mass of brass and iron weighing 57 pounds, there are 15 pounds of iron. Find the ratio of iron to brass.

32. Given the ratio $\frac{2}{3}$ and a positive number x. Prove that $\frac{2+x}{3+x}>\frac{2}{3}$ by subtracting one ratio from the other.

Suggestion. — Proceed as in the proof of Prin. 4, § 118.

PROPORTION

120. Define and illustrate:

1. **Proportion.**
2. **Extremes of a proportion.**
3. **Means of a proportion.**
4. **Mean proportional.**
5. **Third proportional.**
6. **Fourth proportional.**

121. Since a proportion is an equality of ratios each of which may be expressed as a fraction, a proportion may be expressed as an equation each member of which is a fraction.

Hence, it follows that:

General Principle. — *The changes that may be made in a proportion without destroying the equality of its ratios correspond to the changes that may be made in the members of an equation without destroying their equality and in the terms of a fraction without altering the value of the fraction.*

Properties of Proportions

122. Principles. — 1. *In any proportion, the product of the extremes is equal to the product of the means.*

Thus, if $\frac{a}{b}=\frac{c}{d}$, then, $ad = bc$.

It follows that *a mean proportional between two numbers is equal to the square root of their product.*

2. *Either extreme of a proportion is equal to the product of the means divided by the other extreme. Either mean is equal to the product of the extremes divided by the other mean.*

Thus, if $\frac{a}{b}=\frac{c}{d}$, then, $a = \frac{bc}{d}$, $d = \frac{bc}{a}$, $b = \frac{ad}{c}$, and $c = \frac{ad}{b}$.

3. *If the product of two numbers is equal to the product of two other numbers, one pair of them may be made the extremes and the other pair the means of a proportion.*

Thus, if $ad = bc$, then, $\frac{a}{b}=\frac{c}{d}$, or $a : b = c : d$.

4. *If four numbers are in proportion, they are in proportion by* ***alternation.***

Thus, if $\frac{a}{b}=\frac{c}{d}$, then, $\frac{a}{c}=\frac{b}{d}$, or $a:c=b:d$.

5. *If four numbers are in proportion, they are in proportion by* ***inversion.***

Thus, if $\frac{a}{b}=\frac{c}{d}$, then, $\frac{b}{a}=\frac{d}{c}$, or $b:a=d:c$.

6. *If four numbers are in proportion, they are in proportion by* ***composition.***

Thus, if $\frac{a}{b}=\frac{c}{d}$, then, $\frac{a+b}{b}=\frac{c+d}{d}$; also, $\frac{a+b}{a}=\frac{c+d}{c}$.

7. *If four quantities are in proportion, they are in proportion by* ***division.***

Thus, if $\frac{a}{b}=\frac{c}{d}$, then, $\frac{a-b}{b}=\frac{c-d}{d}$; also, $\frac{a-b}{a}=\frac{c-d}{c}$.

8. *If four numbers are in proportion, they are in proportion by* ***composition and division.***

Thus, if $\frac{a}{b}=\frac{c}{d}$, then, $\frac{a+b}{a-b}=\frac{c+d}{c-d}$, or $a+b:a-b=c+d:c-d$.

9. *The products of corresponding terms of any number of proportions form a proportion.*

Thus, if $\frac{a}{b}=\frac{c}{d}$, $\frac{k}{l}=\frac{m}{n}$, and $\frac{x}{y}=\frac{z}{w}$, then, $\frac{akx}{bly}=\frac{cmz}{dnw}$.

EXERCISES

123. Find the value of x in each of the following proportions:

1. $\frac{1}{8}:x=\frac{1}{2}:\frac{4}{5}$.
2. $x:x+1=6:2$.
3. $x-3:x=8:4$.
4. $\frac{x}{12}=\frac{x-12}{3}$.
5. $\frac{3x^2-5x}{6}=\frac{2x^2+3}{4}$.
6. Find a third proportional to $2a$ and $6a$.
7. Find a fourth proportional to $2\frac{1}{2}$, 4, and 8.
8. Find a mean proportional between $4a$ and $9a$.
9. Prove the truth of each principle given in § 122.

When $a : b = c : d$, prove that:

10. $a^2 : b^2 = c^2 : d^2$.

11. $\frac{a}{3} : \frac{b}{3} = \frac{c}{2} : \frac{d}{2}$.

12. $\sqrt{a} : \sqrt{c} = \sqrt{b} : \sqrt{d}$.

13. $a : bc = 1 : d$.

14. $a : c = \frac{1}{d} : \frac{1}{b}$.

15. $a^2 : ab = c^2 : cd$.

16. $\frac{a^2}{b^2} = \frac{ac}{bd}$.

17. $\frac{b^2}{a} = \frac{bd}{c}$.

18. $\frac{a-b}{c-d} = \frac{b}{d}$.

19. $\frac{2c+d}{2a+b} = \frac{d}{b}$.

20. $\frac{2a+3b}{2c+3d} = \frac{3b}{3d}$.

21. $\frac{a^2+b^2}{a^2-b^2} = \frac{c^2+d^2}{c^2-d^2}$.

22. $ma + nb : ma - nb = mc + nd : mc - nd$.

23. $2a + 3c : 2a - 3c = 8b + 12d : 8b - 12d$.

24. $a^3 + a^2b + ab^2 + b^3 : a^3 = c^3 + c^2d + cd^2 + d^3 : c^3$.

25. $a + b + c + d : a - b + c - d = a + b - c - d : a - b - c + d$.

26. If $a : b = c : d$, and if x is a third proportional to a and b, and y a third proportional to b and c, show that the mean proportional between x and y is equal to that between c and d.

Problems

124. 1. Divide \$35 between two men so that their shares shall be in the ratio of 3 to 4.

2. If brass is composed of 2 parts of copper to 1 part of zinc, how much of each substance is required for 75 pounds of brass?

3. A line a inches long was divided into two parts in the ratio $m : n$. Find the length of each part.

4. Two partners gained \$6000 in business one year. Find each one's share, their investments being in the ratio $1 : 4$.

5. Two numbers are in the ratio of 3 to 2. If each is increased by 4, the sums will be in the ratio of 4 to 3. What are the numbers?

SUGGESTION. — Represent the numbers by $3x$ and $2x$.

6. Divide 25 into two parts such that the greater increased by 1 is to the less decreased by 1 as 4 is to 1.

7. Two trains traveled toward each other from two cities 98 miles apart. If their rates of traveling were as 3 is to 4, how many miles did each travel before they met?

8. A man divided his estate of \$50,000 between two heirs in the ratio of 3 to 7. How much did each heir receive?

9. Divide 16 into two parts such that their product is to the sum of their squares as 3 is to 10.

SUGGESTION. — Solve the final equation by factoring.

10. The sum of two numbers is 4, and the square of their sum is to the sum of their squares as 8 is to 5. What are the numbers?

11. A dock is divided into two parts so that the length of the longer is to that of the shorter as 11 is to 6. If its total length is 850 feet, what is the length of each part?

12. The freight earnings of two railroads on a trainload of grain were \$2160. One carried the grain 400 miles, the other 500 miles. Find the earnings apportioned to each road.

13. Find a number that subtracted from each of the numbers 7, 9, 10, and 14 will give four numbers in proportion.

14. What number must be added to each of the numbers 11, 17, 2, and 5 so that the sums shall be in proportion when taken in the order given?

15. If 16 men can do a piece of work in 15 days, how long will it take 20 men to do it?

16. The total receipts of a coal mining company one year were \$16,725,000, and the expenses were to the net earnings as 13 is to 2. What were the expenses? the net earnings?

17. Prove that no four consecutive integers, as n, $n+1$, $n+2$, and $n+3$, can form a proportion.

18. Prove that the ratio of an odd number to an even number, as $2m+1:2n$, cannot be equal to the ratio of another even number to another odd number, as $2x:2y+1$.

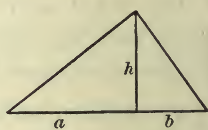
19. The areas of two circles are proportional to the squares of their radii. If the area of a circle is 5 square inches, what is the area of a circle whose radius is twice the radius of the first circle?

20. The formula $pd = WD$ expresses the physical law that, when a **lever** just balances, the product of the numerical measures of the power (p) and its distance (d) from the **fulcrum** (F) is equal to the product of the numerical measures of the weight (W) and its distance (D) from the fulcrum. Express this law by means of a proportion.

21. Solve the proportion obtained in exercise 20 for W and find what weight a power of 60 pounds will support by means of a lever, if $d = 8$ feet and $D = 3$ feet.

22. A pressure of 50 pounds was exerted upon one end of a 5-foot bar to balance a weight of 200 pounds at the other end of the bar. How far was the weight from the fulcrum?

23. A farmer has a team, one horse of which weighs 1200 pounds and the other 1400 pounds. If draft power is proportional to weight, where shall he put the clevis (fulcrum) on his 50-inch double-tree (lever)?



24. In the figure (right triangle) the altitude h is a mean proportional between the segments a and b of the hypotenuse. Find the length of b, if $h = 8$ and $a = 10$.

25. The following is a simple relation for pulleys belted together: The speed (S), revolutions made per minute, of the driving pulley is to speed (s) of the driven pulley as the diameter (d) of the driven pulley is to the diameter (D) of the driving pulley. Write the proportion, using the letters S, s, D, and d.

26. What is the speed of a driving pulley 10 inches in diameter, if the driven pulley is 12 inches in diameter and its speed is 500 revolutions per minute?

SIMULTANEOUS SIMPLE EQUATIONS

TWO UNKNOWN NUMBERS

125. Define and illustrate:

1. **Indeterminate equation.**
2. **Dependent equations.**
3. **Independent equations.**
4. **Consistent equations.**
5. **Inconsistent equations.**
6. **Elimination.**

126. PRINCIPLE. — *Any single equation involving two or more unknown numbers is indeterminate.*

Elimination by Addition or Subtraction

EXERCISES

127. 1. Solve the equations $3x + 2y = 12$ and $2x + 3y = 13$.

SOLUTION $6 + 6$

$$\begin{cases} 3x + 2y = 12, & (1) \\ 2x + 3y = 13. & (2) \end{cases}$$

Multiply (1) by 3, $\quad 9x + 6y = 36. \quad (3)$

Multiply (2) by 2, $\quad 4x + 6y = 26. \quad (4)$

Subtract (4) from (3), $\quad 5x = 10. \quad (5)$

$\therefore x = 2. \quad (6)$

Substitute (6) in (1), $\quad 6 + 2y = 12;$

whence, $\quad y = 3.$

To *verify*, substitute 2 for x and 3 for y in each given equation.

RULE. — *If necessary, multiply or divide the equations by such numbers as will make the coefficients of the quantity to be eliminated numerically equal.*

Eliminate by addition if the resulting coefficients have unlike signs, or by subtraction if they have like signs.

Solve by addition or subtraction, and verify results:

2. $\begin{cases} 4x - 5y = 3, \\ 3x + 5y = 11. \end{cases}$

3. $\begin{cases} x + y = 8, \\ 2x + 3y = 19. \end{cases}$

4. $\begin{cases} 3u - v = 4, \\ u + 3v = -2. \end{cases}$

5. $\begin{cases} 4x - 3y = 5, \\ 5x - 6y = 5\frac{1}{2}. \end{cases}$

6. $\begin{cases} 4x + y = 25, \\ 2y - 5x = 24. \end{cases}$

7. $\begin{cases} 3x - y = 14, \\ 2x + 2y = 28. \end{cases}$

8. $\begin{cases} x + 5y = 7, \\ 4x + 3y = 11. \end{cases}$

9. $\begin{cases} 7s - 9v = 6, \\ s + 2v = 14. \end{cases}$

10. $\begin{cases} 3x - 2y = -\frac{1}{2}, \\ 2x + 5y = 6. \end{cases}$

11. $\begin{cases} 5x + 2y = 16, \\ 3x - 5y = -9. \end{cases}$

12. $\begin{cases} 7a - 3b = 9, \\ 3a - 2b = 1. \end{cases}$

13. $\begin{cases} 5u + 9v = 60, \\ 4u - 4v = -8. \end{cases}$

14. $\begin{cases} 2s + 8t = 7, \\ 7s + 12t = 12\frac{1}{2}. \end{cases}$

15. $\begin{cases} 7x - 4y = 81, \\ 5x + 3y = 52. \end{cases}$

Elimination by Substitution

EXERCISES

128. **1.** Solve the equations $3x + \frac{1}{2}y = 8$ and $5x - y = 6$.

Solution.
$$\begin{cases} 3x + \frac{1}{2}y = 8, & (1) \\ 5x - \phantom{\frac{1}{2}}y = 6. & (2) \end{cases}$$

Solve (2) for y, $\quad y = 5x - 6. \quad (3)$

Substitute the value of y from (3) in (1),

$$3x + \tfrac{1}{2}(5x - 6) = 8. \quad (4)$$

Solve (4), $\quad x = 2. \quad (5)$

Substitute (5) in (3), $\quad y = 10 - 6 = 4.$

Rule. — *Find an expression for the value of either of the unknown numbers in one of the equations.*

Substitute this value for that unknown number in the other equation, and solve the resulting equation.

Solve by substitution, and verify results:

2. $\begin{cases} x+y=6, \\ 2x+y=10. \end{cases}$

3. $\begin{cases} x-y=-1, \\ 2x+3y=18. \end{cases}$

4. $\begin{cases} 3x-4y=14, \\ x-4=2y. \end{cases}$

5. $\begin{cases} 2s+4t=20, \\ 3s-5t=-3. \end{cases}$

6. $\begin{cases} 5x-y=5, \\ 3x-2y=-4. \end{cases}$

7. $\begin{cases} 3x-12y=30, \\ x+6y=15. \end{cases}$

8. $\begin{cases} 25=5a-b, \\ 28=3a+2b. \end{cases}$

9. $\begin{cases} 4s+3t=3, \\ 5t-3s=34. \end{cases}$

10. $\begin{cases} 2x=4y+14, \\ 3x-7y=23. \end{cases}$

11. $\begin{cases} 7x-5y=15, \\ 3x+3y=9. \end{cases}$

12. $\begin{cases} 2x-3y=-7, \\ 4x-5y=-9. \end{cases}$

13. $\begin{cases} 2-x=4y, \\ 3y-10=2(2-x). \end{cases}$

MISCELLANEOUS EXERCISES

129. Solve and test, eliminating before or after clearing of fractions as may be more advantageous:

1. $\begin{cases} \dfrac{x}{3}=11-\dfrac{y}{2}, \\ \dfrac{x}{3}+\dfrac{2y}{7}=8. \end{cases}$

2. $\begin{cases} \dfrac{x}{3}=\dfrac{y}{2}, \\ \dfrac{x}{3}-\dfrac{y}{3}=1. \end{cases}$

3. $\begin{cases} \dfrac{3x}{4}+\dfrac{2y}{3}=20, \\ \dfrac{x}{2}+\dfrac{3y}{4}=17. \end{cases}$

4. $\begin{cases} \dfrac{x}{2}-\dfrac{y}{3}-1=0, \\ \dfrac{2x-1}{2}-\dfrac{3y-1}{3}=\dfrac{5}{6}. \end{cases}$

5. $\begin{cases} \dfrac{1}{x-1}-\dfrac{3}{x+y}=0, \\ \dfrac{3}{x-y}+3=0. \end{cases}$

6. $\begin{cases} \dfrac{x}{2}-12=\dfrac{y+32}{4}, \\ \dfrac{y}{8}+\dfrac{3x-2y}{5}=25. \end{cases}$

Solve, and test each result:

7. $\begin{cases} \dfrac{x-1}{4}+y=3, \\ \dfrac{x-1}{4}+4y=9. \end{cases}$

8. $\begin{cases} \dfrac{x}{8}+4y=15, \\ \dfrac{x}{6}+\dfrac{2y}{3}=6. \end{cases}$

9. $\begin{cases} \dfrac{7+x}{5}-\dfrac{2x-y}{4}=3y-5, \\ \dfrac{5y-7}{2}+\dfrac{4x-3}{6}=18-5x. \end{cases}$

10. $\begin{cases} \dfrac{.2y+.5}{1.5}=\dfrac{.49x-.7}{4.2}, \\ \dfrac{.5x-.2}{1.6}=\dfrac{41}{16}-\dfrac{1.5y-11}{8}. \end{cases}$

Solve the following as if $\dfrac{1}{x}$ and $\dfrac{1}{y}$ were the unknown numbers, and then find the values of x and y:

11. $\begin{cases} \dfrac{2}{x}+\dfrac{4}{y}=10, \\ \dfrac{6}{x}-\dfrac{2}{y}=10. \end{cases}$

12. $\begin{cases} \dfrac{7}{x}+\dfrac{8}{y}=30, \\ \dfrac{7}{y}+\dfrac{8}{x}=30. \end{cases}$

13. $\begin{cases} \dfrac{5}{x}+\dfrac{6}{y}=7, \\ \dfrac{7}{x}+\dfrac{9}{y}=10. \end{cases}$

14. $\begin{cases} \dfrac{7}{8x}-\dfrac{2}{3y}=10, \\ \dfrac{5}{6x}-\dfrac{2}{11y}=17. \end{cases}$

Solve the following as if $\dfrac{1}{x-1}$, $\dfrac{1}{y+1}$, etc., were the unknown numbers, and then find the values of x and y:

15. $\begin{cases} \dfrac{1}{x-1}+\dfrac{1}{y+1}=5, \\ \dfrac{2}{x-1}+\dfrac{3}{y+1}=12. \end{cases}$

16. $\begin{cases} \dfrac{5}{x-1}-\dfrac{3}{y-1}=14, \\ \dfrac{2}{x-1}-\dfrac{1}{y-1}=6. \end{cases}$

17. $\begin{cases} \dfrac{1}{y}=\dfrac{3}{2-x}, \\ \dfrac{5}{y}=\dfrac{6}{2-x}+9. \end{cases}$

18. $\begin{cases} \dfrac{4}{x}=\dfrac{1}{y+3}, \\ \dfrac{7}{x}=\dfrac{3}{y+3}-10. \end{cases}$

Literal Simultaneous Equations

130. In solving literal simultaneous equations, elimination is performed usually by addition or subtraction for each unknown number.

EXERCISES

131. Solve for x and y, and test as on page 77:

1. $\begin{cases} ax + by = m, \\ ax - by = n. \end{cases}$

2. $\begin{cases} a^2x + cy = 2, \\ y - cx = 1. \end{cases}$

3. $\begin{cases} ax + by = r, \\ ax + cy = s. \end{cases}$

4. $\begin{cases} bx + cy = 2, \\ \dfrac{x}{d} - \dfrac{y}{c} = \dfrac{1}{cd}. \end{cases}$

5. $\begin{cases} \dfrac{x}{a} + \dfrac{y}{b} = 1, \\ \dfrac{x}{b} - \dfrac{y}{a} = \dfrac{1}{2}. \end{cases}$

6. $\begin{cases} 2x + ay = b, \\ ax + 2y = c. \end{cases}$

7. $\begin{cases} ax - dy = b, \\ mx - ny = b. \end{cases}$

8. $\begin{cases} ax + by = c, \\ bx - ay = d. \end{cases}$

9. $\begin{cases} x + y = ab(a + b), \\ \dfrac{x}{a} + \dfrac{y}{b} = 2ab. \end{cases}$

10. $\begin{cases} \dfrac{a}{x} + \dfrac{b}{y} = c, \\ \dfrac{m}{x} + \dfrac{n}{y} = e. \end{cases}$

11. Given $\begin{cases} F = Ma, \\ s = \frac{1}{2}at^2. \end{cases}$

Find the values of F and a when $M = 15$, $s = 72$, and $t = 6$.

12. Given $\begin{cases} l = a + (n - 1)d, \\ s = \dfrac{n}{2}(a + l). \end{cases}$

Find the values of a and l when $n = 50$, $d = 2$, and $s = 2500$; the values of d and a when $l = 50$, $n = 25$, and $s = 650$.

13. Given $\begin{cases} l = ar^{n-1}, \\ s = \dfrac{rl - a}{r - 1}. \end{cases}$

Find the values of a and l when $r = 2$, $n = 11$, and $s = 2047$.

THREE OR MORE UNKNOWN NUMBERS

132. PRINCIPLE. — *Every system of independent simultaneous simple equations involving the same number of unknown numbers as there are equations can be solved, and is satisfied by one and only one set of values of its unknown numbers.*

EXERCISES

133. 1. Solve the equations $\begin{cases} 2x - 3y - z = 2, & (1) \\ 3x + y - 2z = 8, & (2) \\ x - 2y + 3z = 4. & (3) \end{cases}$

SUGGESTION. — Eliminate z from (1) and (2) by subtraction and from (1) and (3) by addition; then solve the resulting equations.

RULE. — *Eliminate one unknown number from any convenient pair of equations, and the same number from a different pair. Solve the resulting equations.*

Solve, and test all results:

2. $\begin{cases} x + y + z = 18, \\ x - y + z = 6, \\ x + y - z = 4. \end{cases}$

3. $\begin{cases} x - 2y + 2z = 5, \\ 2x - y + z = 7, \\ x + 2y + 2z = 21. \end{cases}$

4. $\begin{cases} v + x - y = 2, \\ v - x + y = 4, \\ x - v + y = 8. \end{cases}$

5. $\begin{cases} x + y = 9, \\ y + z = 7, \\ z + x = 5. \end{cases}$

6. $\begin{cases} 4x - 5y + 3z = 14, \\ x + 7y - z = 13, \\ 2x + 5y + 5z = 36. \end{cases}$

7. $\begin{cases} x + 3y + z = 14, \\ x + y + 3z = 16, \\ 3x + y + z = 20. \end{cases}$

SUGGESTION. — In exercise 4, subtract each equation from the sum of the equations.

8. $\begin{cases} v + x + y = 15, \\ x + y + z = 18, \\ y + z + v = 17, \\ z + v + x = 16. \end{cases}$

9. $\begin{cases} y + z + v - x = 22, \\ z + v + x - y = 18, \\ v + x + y - z = 14, \\ x + y + z - v = 10. \end{cases}$

SUGGESTION. — In exercise 8, subtract each equation from $\frac{1}{3}$ of the sum of the equations.

Solve for x, y, z, and v:

10. $\begin{cases} axy - x - y = 0, \\ bzx - z - x = 0, \\ cyz - y - z = 0. \end{cases}$

11. $\begin{cases} x + y - z = 0, \\ x - y = 2b, \\ x + z = 3a + b. \end{cases}$

12. $\begin{cases} v + x = 2a, \\ x + y = 2a - z, \\ y + z = a + b, \\ v - z = a + c. \end{cases}$

13. $\begin{cases} abxyz + cxy - ayz - bzx = 0, \\ bcxyz + ayz - bzx - cxy = 0, \\ caxyz + bzx - cxy - ayz = 0. \end{cases}$

14. $\begin{cases} x + y + z = a + b + c, \\ x + 2y + 3z = b + 2c, \\ x + 3y + 4z = b + 3c. \end{cases}$

15. $\begin{cases} v + x + y = a + 2b + c, \\ x + y + z = 3b, \\ y + z + v = a + b, \\ z + v + x = a + 3b - c. \end{cases}$

Problems

134. To solve a problem by means of a statement involving two or more unknown numbers, *there must be as many given conditions and as many equations as there are unknown numbers.*

Solve and verify the following problems.

Find two numbers related to each other as follows:

1. Sum $= 14$; difference $= 8$.

2. Sum of 2 times the first and 3 times the second $= 34$; sum of 2 times the first and 5 times the second $= 50$.

3. Sum $= 18$; sum of the first and 2 times the second $= 20$.

4. The difference between two numbers is 4 and $\frac{1}{4}$ of their sum is 9. Find the numbers.

5. New York once owned 186 parks. Of these the number that had an area of less than one acre was 28 less than the number of the larger ones. Find the number of small parks.

6. In Dawson, Alaska, recently, 2 tons of coal and 3 cords of wood cost together \$ 68. If 3 tons of coal cost the same as 4 cords of wood, what was the cost of a ton of coal? of a cord of wood?

7. The sum of 3 numbers is 162. The quotient of the second divided by the first is 2; of the third divided by the first is 3. Find the numbers.

8. A merchant has 100 bills valued at \$275. Some are 2-dollar bills and the rest 5-dollar bills. How many bills of each kind has he?

9. A paymaster has 110 coins valued at \$40. Some are quarters and the remainder half dollars. How many coins has he of each?

10. In a plum orchard of 133 trees, the number of Lombard trees is 7 more than $\frac{5}{9}$ of the number of Gage trees. Find the number of each kind.

11. If 5 pounds of sugar and 8 pounds of coffee cost \$2.70, and at the same price 9 pounds of sugar and 12 pounds of coffee cost \$4.14, how much does each cost per pound?

12. A lieutenant of the U.S. navy, receiving \$1620 yearly, earned \$150 a month while on sea duty and \$127.50 a month while on shore duty. How many months was he on land?

13. A farmer bought 80 acres of land for \$4500. If part of it cost \$60 per acre and the remainder $\frac{1}{3}$ as much per acre, how many acres did he buy at each price?

14. If 8 baskets and 4 crates together hold 8 bushels of tomatoes, and 6 baskets and 8 crates together hold $9\frac{3}{4}$ bushels, what is the capacity of a basket? of a crate?

15. If 2 is added to the numerator of a certain fraction, the value of the fraction becomes $\frac{3}{5}$; if 1 is subtracted from the denominator, the value becomes $\frac{1}{2}$. What is the fraction?

SUGGESTION. — Let $\frac{x}{y}$ = the fraction.

16. The sum of two fractions whose numerators are 3, is 3 times the smaller; 3 times the smaller subtracted from twice the larger gives $\frac{3}{8}$. What are the fractions?

17. The sum of the digits in a number of two figures is 9 and their difference is 3. Find the number. (Two answers.)

18. The sum of the digits of a two-digit number is 5. If the number is multiplied by 3, and 1 is taken from the result, the digits are reversed. Find the number.

Suggestion. — The sum of x tens and y units is $(10x + y)$ units; of y tens and x units, $(10y + x)$ units.

19. The sum of the two digits of a certain number is 12, and the number is 2 less than 11 times its tens' digit. What is the number?

20. If a certain number of two digits is divided by their sum, the quotient is 8; if 3 times the units' digit is taken from the tens' digit, the result is 1. Find the number.

21. Separate 800 into three parts, such that the sum of the first, $\frac{1}{2}$ of the second, and $\frac{2}{5}$ of the third is 400; and the sum of the second, $\frac{3}{4}$ of the first, and $\frac{1}{4}$ of the third is 400.

22. A certain number is expressed by three digits whose sum is 14. If 693 is added to the number, the digits will appear in reverse order. If the units' digit is equal to the tens' digit increased by 6, what is the number?

23. If 10 pounds of chicken feathers and 6 pounds of duck feathers cost \$2.43, and 16 pounds of the former and 5 pounds of the latter cost \$2.37, what is the cost per pound of each kind of feathers?

24. A 5-dollar gold piece weighs $\frac{1}{2}$ as much as a 10-dollar gold piece. If the combined weight of 3 of the former and 2 of the latter is 903 Troy grains, what is the weight of each?

25. If Rio coffee costs 20¢ per pound and Java coffee, 32¢ per pound, how many pounds of each must be bought to fill a 120-pound canister making a blend worth 28¢ per pound?

26. If a bushel of corn is worth r cents, and a bushel of wheat is worth s cents, how many bushels of each must be mixed to make a bushels worth b cents per bushel?

27. If a rectangular floor were 2 feet wider and 5 feet longer, its area would be 140 square feet greater. If it were 7 feet wider and 10 feet longer, its area would be 390 square feet greater. What are its dimensions?

28. The cost of cooking meat for 1 hour averages 2.128¢ less by gas than by electricity. If meat can be cooked 4 hours by the former means for .256¢ less than it can be cooked 2 hours by the latter, what is the cost of each per hour?

29. To burn weeds along a railroad by a gasoline burner costs $16.66 less per mile than to cut them by hand. It costs as much to clear 160 miles by the former method as 41 miles by the latter. Find the cost per mile by each method.

30. Single yarn of imitation silk is put up in three qualities, A, B, and C. 5 pounds of A and 2 pounds of B cost $8.64; 3 pounds of B and 1 pound of C cost $5.40; 2 pounds of A and 3 pounds of C cost $6.72. Find the cost per pound of each quality.

31. The winning baseball team of the National League one year won 44 games more than it lost. If the number won had been 8 less and the number lost 8 more, the ratio of the former to the latter would have been 13 : 9. Find the number of games won; the number of games lost.

32. A boatman trying to row up a river drifted back at the rate of 2 miles an hour, but he could row down the river at the rate of $12\frac{1}{2}$ miles an hour. Find the rate of the current.

33. A takes 3 hours longer than B to walk 30 miles, but if A doubles his pace, he takes 2 hours less than B. Find A's rate; B's rate.

34. A and B can do a piece of work in 10 days; A and C can do it in 8 days; and B and C can do it in 12 days. How long will it take each to do it alone?

35. A and B can do a piece of work in r days; A and C can do it in s days; and B and C can do it in t days. How long will it take each to do it alone?

36. When weighed in water silver loses .095 of its weight and gold .051 of its weight. If an alloy of gold and silver weighing 12 ounces loses .788 of an ounce when weighed in water, how many ounces of each are there in the piece?

37. When weighed in water tin loses .137 of its weight and copper .112 of its weight. If an alloy of tin and copper weighing 18 pounds loses 2.316 pounds when weighed in water, how many pounds of each are there in the piece?

38. When weighed in water tin loses .137 of its weight and lead loses .089 of its weight. If an alloy of tin and lead weighing 14 pounds loses 1.594 pounds when weighed in water, how many pounds of each are there in the piece?

39. Two pumps are discharging water into a tank. If the first works 5 minutes and the second 3 minutes, they will pump 2260 gallons of water; if the first works 4 minutes and the second 7 minutes, they will pump 3280 gallons. Find their capacity per minute.

40. A and B together can do a piece of work in 12 days. After A has worked alone for 5 days, B finishes the work in 26 days. In what time can each alone do the work?

41. If 4 boys and 6 men can do a piece of work in 30 days, and 5 boys and 5 men can do the same work in 32 days, how long will it take 12 men to do the work?

42. A and B can do a piece of work in a days, or if A works m days alone, B can finish the work by working n days. In how many days can each do the work?

43. A and B can do a piece of work in a days; A works alone m days, when A and B finish it in n days. In how many days can each do it alone?

44. A can build a wall in c days, and B can build it in d days. How many days must each work so that, after A has done a part of the work, B can take his place and finish the wall in a days from the time A began?

45. At simple interest a sum of money amounted to \$2472 in 9 months and to \$2528 in 16 months. Find the amount of money at interest and the rate.

46. Mr. Shaw invested \$8025, a part at $3\frac{1}{2}\%$ and the rest at 4%. If the annual income from both investments was \$309, what was the amount of each investment?

47. A man invested a dollars, a part at r per cent and the rest at s per cent yearly. If the annual income from both investments was b dollars, what was the amount of each investment?

48. A sum of money at simple interest amounted to b dollars in t years, and to a dollars in s years. What was the principal, and what was the rate of interest?

49. A certain number of people charter an excursion boat, agreeing to share the expense equally. If each pays a cents, there will be b cents lacking from the necessary amount; and if each pays c cents, d cents too much will be collected. How many persons are there, and how much should each pay?

50. A mine is emptied of water by two pumps which together discharge m gallons per hour. Both pumps can do the work in b hours, or the larger can do it in a hours. How many gallons per hour does each pump discharge? What is the discharge of each per hour when $a=5$, $b=4$, and $m=1250$?

51. Two trains are scheduled to leave A and B, m miles apart, at the same time, and to meet in b hours. If the train that leaves B is a hours late and runs at its customary rate, it will meet the first train in c hours. What is the rate of each train? What is the rate of each, if $m=800$, $c=9$, $a=1\frac{3}{5}$, and $b=10$?

52. A man ordered a certain amount of cement and received it in c barrels and d bags; a barrels and b bags made $\frac{m}{n}$ of the total weight. How many barrels or how many bags alone would have been needed? Find the number of each, if $c=16$, $d=15$, $a=6$, $b=15$, $m=1$, and $n=2$.

GRAPHIC SOLUTIONS

LINEAR FUNCTIONS

function notation

135. An expression involving one or more letters is called a **function** of those letters.

Thus, $3x - 2$ is a function of x; also $x + y$ is a function of x and y.

Again, the area of a rectangle is a function of its base and altitude, $A = bh$; percentage is a function of the base and rate, $p = br$.

136. The *symbol* for any given function of x is $f(x)$, read "function of x." Other functions of x in the same discussion may be represented, if desired, by $F(x)$, $f'(x)$, etc., read "large F function of x," "f-prime function of x," etc.

Values of $f(x)$ corresponding to particular values of x, as 1, 2, 0, etc., are usually indicated by $f(1)$, $f(2)$, $f(0)$, etc., respectively.

Thus, if $f(x) = 6x + 9$, $f(1) = 6 + 9 = 15$; $f(2) = 12 + 9 = 21$; $f(0) = 9$.

137. A quantity whose value changes in the same discussion is called a **variable**; a quantity whose value remains the same is a **constant**.

Thus, in the formula for the volume of a sphere, $V = \frac{4}{3}\pi r^3$, the volume changes for changing values of r; then V and r are *variables*, but π, whose value remains the same whatever the value of r, is a *constant*.

EXERCISES

138. 1. Evaluate $f(x) = 2x - 7$ for $x = 1$; for $x = 3$; for $x = 0$.

2. When $f(x) = 2(x - \frac{3}{2})$, find $f(1)$; $f(2)$; $f(5)$; $f(8)$.

3. When $f(x) = \frac{2}{3}(3 - x)$, find $f(0)$; $f(3)$; $f(6)$; $f(12)$.

4. When $F(x) = \frac{1}{2}(5 - x)$, find $F(4)$; $F(1)$; $F(0)$; $F(7)$.

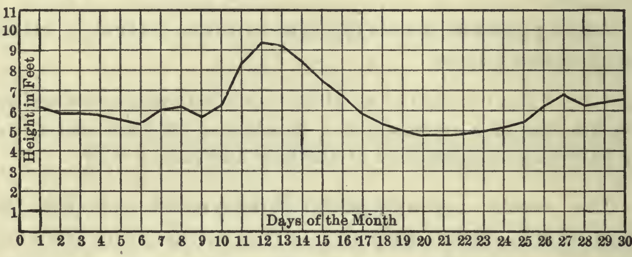
5. When $f(y) = 3(2 - y)$, find $f(0)$; $f(3)$; $f(15)$; $f(20)$.

6. Evaluate $f(u) = \frac{2}{3}(u + 8)$ for $u = 1$; for $u = 7$; for $u = 16$.

7. When $f'(x) = .7(x + 1.5)$, find $f'(0)$; $f'(2.5)$; $f'(\frac{3}{2})$; $f'(\frac{9}{2})$.

139. Graphical representation. — When related varying quantities in a series are to be compared, it is often convenient and very effective to represent them by a diagram, or **graph**.

The following graph represents the height of water in a certain river above 0 of the gauge from daily observations during the month of September.



The horizontal distances represent *time in days* and the vertical distances, the *height* of water *in feet.*

Thus, on the 19th day of the month the height of the water is represented by the vertical line drawn upward from 19 and is 5 feet. In fact, every point of the irregular black line, or *graph*, exhibits a pair of corresponding values of the two related quantities — *days* and *height* of water.

From the graph answer the following:

1. How high was the water on Sept. 1? on Sept. 23?
2. On what day of the month was the water highest? lowest?
3. What was the maximum height? the minimum height? the range between them?
4. What part of the month shows the most rapid changes?
5. Give the time of the greatest change in a single day.

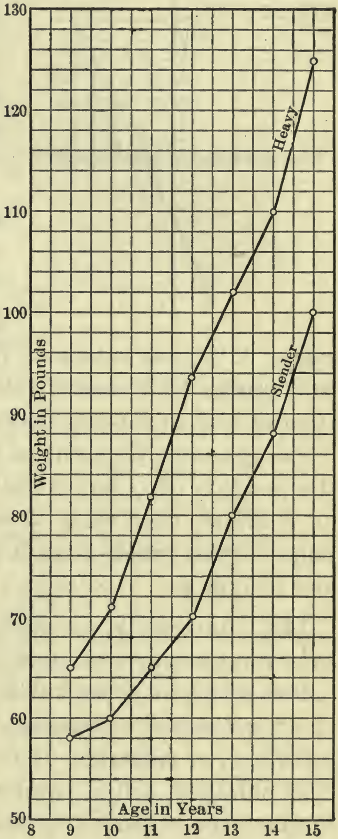
Graphs have very many uses. The statistician uses them to present information in a telling way. The broker and the merchant use them to compare the rise and fall of prices. The physician uses them to record the progress of diseases. The engineer uses them in testing materials and in computing. The scientist uses them in his investigations of the laws of nature. In short, graphs may be used whenever two related quantities are to be compared throughout a series of values.

The use of paper ruled in small squares, called squared paper or coördinate paper, is advised in plotting graphs.

140. The two graphs given on this page present to the eye the comparative weights of two standard types ("slender" and "heavy") of boys between the ages of 9 years and 15 years.

The scales to which these graphs are constructed are, for vertical distances, 1 space represents 2 pounds, and for horizontal distances, 2 spaces represent 1 year. The vertical spaces for 0 pounds to 49 pounds, and the horizontal spaces for 0 years to 7 years are omitted.

Graphs may be constructed to any convenient scale and, if desired, the horizontal scale may differ from the vertical scale.



1. From the graph read the standard weight of the slender type of boy at age 9; at age 10; at age 11; at age 12; at age 13; at age 14; at age 15.

2. Read the standard weight of the heavy type of boy for each age from 9 to 15 inclusive.

3. During what year does each type increase in weight most rapidly? least rapidly?

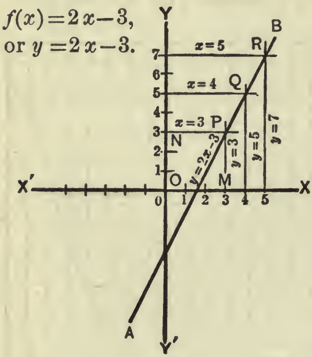
4. What is the difference in weight of the two types at age 9? at age 10? at age 11? at 12? at 13? at 14? at 15?

5. At what age is the difference in weight greatest? least?

6. What is the weight of the slender type at $9\frac{1}{2}$ years? at $12\frac{1}{2}$ years? at $14\frac{1}{2}$ years? of the heavy type at $13\frac{1}{2}$ years?

7. What is the approximate age of the slender type of boy when he weighs 72 pounds? 97 pounds? of the heavy type when he weighs 90 pounds? 98 pounds?

141. Let $f(x) = 2x - 3$. It is evident that we may give x a series of values, obtaining a corresponding series of values of $f(x)$, and that *the number of pairs of values of x and $f(x)$ is unlimited.* All these values of x and $f(x)$ may be represented by a graph, just as in the preceding illustrations the corresponding values of two variables were represented by a graph.



The line AB is the *graph* of the *function* $2x - 3$ or of the corresponding equation $y = 2x - 3$.

Values of x are represented by lines laid off on or parallel to an ***x*-axis**, $X'X$, and values of $f(x)$ by lines laid off on or parallel to a ***y*-axis**, $Y'Y$ (usually drawn perpendicular to the x-axis), the function of x being denoted by y.

For example, the position of P shows that when $x = 3$, $y = 3$; the position of Q shows that when $x = 4$, $y = 5$; the position of R shows that when $x = 5$, $y = 7$; etc. Evidently every point of the graph gives a pair of corresponding values of x and $f(x)$, or y.

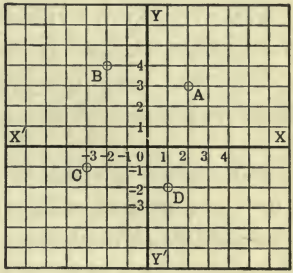
142. Conversely, to locate any point with reference to two axes for the purpose of representing a pair of corresponding values of x and y, the value of x may be laid off on the x-axis as an *x-distance*, or **abscissa**, and that of y on the y-axis as a *y-distance*, or **ordinate**. If from each of the points on the axes thus obtained, a line parallel to the other axis is drawn, the intersection of these two lines locates the point.

Thus, to represent the corresponding values $x = 3$, $y = 3$, a point P may be located by measuring 3 units from O to M on the x-axis and 3 units from O to N on the y-axis, and then drawing a line from M parallel to OY, and one from N parallel to OX, producing these lines until they intersect.

143. The abscissa and ordinate of a point referred to two perpendicular axes are called its **rectangular coördinates**.

Plotting Points and Constructing Graphs

144. By custom *positive* values of x are laid off from the zero-point, or **origin**, toward the *right*, and *negative* values toward the *left*. Also *positive* values of y are laid off *upward* and *negative* values *downward*.



The point A in the figure may be designated as 'the point (2, 3),' or by the equation $A = (2, 3)$.

Similarly, $B = (-2, 4)$, $C = (-3, -1)$, and $D = (1, -2)$.

The abscissa is always written first.

EXERCISES

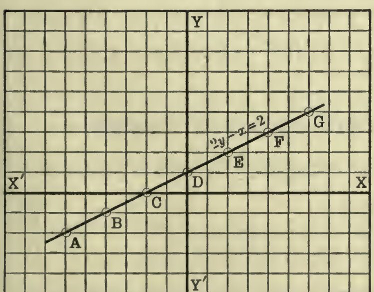
145. Draw two axes at right angles and locate these points:

1. $A = (2, 4)$.
2. $B = (-3, 2)$.
3. $C = (1, -2)$.
4. $D = (-1, -1)$.
5. $(0, -4)$.
6. $(-2, 0)$.
7. $(10, 8)$.
8. $(-5, 11)$.
9. $(-4, -6)$.
10. $(12, -9)$.
11. $(-6, 12)$.
12. $(-7, -8)$.

13. Construct the graph of the equation $2y - x = 2$.

SOLUTION. — Solving for y, we have $y = \frac{1}{2}(x+2)$, in which we substitute values for x and determine corresponding values of y as tabulated below.

The points whose coördinates are given in the table are then plotted.



x	y	POINT
−6	−2	A
−4	−1	B
−2	0	C
0	1	D
2	2	E
4	3	F
6	4	G

A line drawn through A, B, C, D, etc., is the graph of $2y - x = 2$.

Construct the graph of:

14. $f(x)=x-3$.	16. $f(x)=2x+3$.	18. $2y=x$.
15. $f(x)=2-x$.	17. $y=2-3x$.	19. $x+2y=-6$.

146. It is now evident that *the graph of a simple equation in two unknown numbers is a straight line.*

For this reason a simple equation is sometimes called a **linear equation**, and the corresponding function, a **linear function.**

147. Since a straight line is determined by two points, to plot the graph of a linear equation, *plot two points and draw a straight line through them.* To find where the graph intersects the x-axis, let $y=0$; to find where it intersects the y-axis, let $x=0$.

Thus, in $y=\frac{1}{2}(x+2)$, page 111, when $y=0$, $x=-2$, locating C; when $x=0$, $y=1$, locating D. Draw a straight line through C and D.

If the points plotted as just illustrated are near together, for the *sake of accuracy* plot points farther apart. In any case *check* the work by plotting a third point and determining whether it lies on the graph.

EXERCISES

148. Construct the graph of:

1. $y=x-1$.	6. $2x-5y=10$.	11. $5x-y=2\frac{1}{2}$.
2. $y-2x=2$.	7. $4x+3y=12$.	12. $2x-\frac{1}{2}y=-2$.
3. $3y+x=6$.	8. $6+3x=2y$.	13. $\frac{1}{3}x-\frac{1}{2}y=3$.
4. $3x-y=9$.	9. $3x+6y=0$.	14. $.2x+.5y=1$.
5. $x+2y=-8$.	10. $2x-y-4=0$.	15. $.4x+.6y=-.8$.

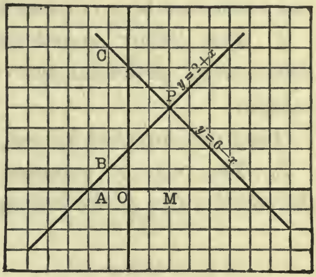
Graphic Solutions of Simultaneous Linear Equations

149. Let it be required to solve graphically the equations

$$\begin{cases} y=2+x, & (1) \\ y=6-x. & (2) \end{cases}$$

Using the same axes, we construct the graph of each equation as shown on the next page.

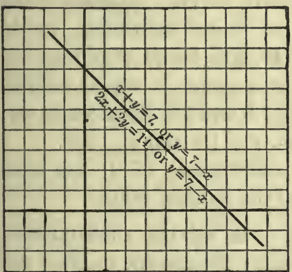
We desire to discover for what values of x and y *both* equations are satisfied.



When $x = -1$, $y = AB = 1$ in (1) and $AC = 7$ in (2). Similarly, the values of y in the two equations differ for every value of x except $x = 2$; when $x = 2$, $y = MP = 4$ in *both* equations.

The required values of x and y, then, are represented graphically by the coördinates of P, *the intersection of the graphs.*

150. Let the given equations be



$$\begin{cases} x + y = 7, & (1) \\ 2x + 2y = 14. & (2) \end{cases}$$

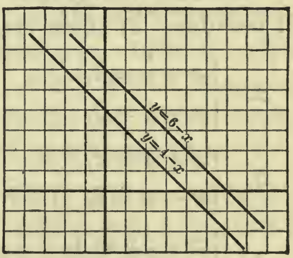
If we try to eliminate either x or y, we find that (1) and (2) are just alike.

Since $y = 7 - x$ in both (1) and (2), the values of y are the same for each value of x.

The graphic analysis, like the algebraic analysis, shows that the equations are **indeterminate**, for *their graphs coincide.*

151. Let the given equations be

$$\begin{cases} y = 6 - x, & (1) \\ y = 4 - x. & (2) \end{cases}$$



For every value of x the values of y in (1) and (2) differ by 2, and the graphs are 2 units apart vertically.

In algebraic language, the equations cannot be simultaneous, that is, they are **inconsistent**. In graphical language, their graphs *cannot intersect*, being parallel straight lines.

152. Principles. — 1. *A single linear equation involving two unknown numbers is indeterminate.*

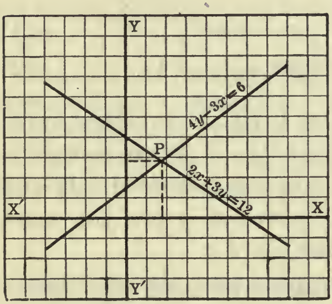
2. *Two linear equations involving two unknown numbers are determinate, provided the equations are independent and simultaneous.*

They are satisfied by one, and only one, pair of common values.

3. *The pair of common values is represented graphically by the coördinates of the intersection of their graphs.*

EXERCISES

153. **1.** Solve graphically the equations $\begin{cases} 4y - 3x = 6, \\ 2x + 3y = 12. \end{cases}$



Solution. — On plotting the graphs of both equations, as in §§ 145–148, it is found that they intersect at a point P, whose coördinates are 1.8 and 2.8, approximately.

Hence, $x = 1.8$ and $y = 2.8$.

The coördinates of P are estimated to the nearest tenth.

Note. — In solving simultaneous equations by the graphic method the same axes must be used for the graphs of both equations.

Construct the graphs of each system of equations. Solve, if possible. If there is no solution, tell why.

2. $\begin{cases} x + y = 4, \\ x - y = 2. \end{cases}$

3. $\begin{cases} x + y = 2, \\ y - x = 6. \end{cases}$

4. $\begin{cases} x + 2y = 4, \\ 2x - y = 3. \end{cases}$

5. $\begin{cases} x = y + 2, \\ x = y - 3. \end{cases}$

6. $\begin{cases} 3x + 2y = 7, \\ 2y - x = 3. \end{cases}$

7. $\begin{cases} 2x = 3 + y, \\ 2y = 4x - 6. \end{cases}$

8. $\begin{cases} 3y + 2x = 4, \\ 3x + 2y = 1. \end{cases}$

9. $\begin{cases} 2y = 3x, \\ x + 4y = 14. \end{cases}$

Solve graphically as instructed on page 114:

10. $\begin{cases} 4y = 10 - 2x, \\ y - 2x = -2. \end{cases}$

11. $\begin{cases} 4x + 3y = 14, \\ 2x - y = 0. \end{cases}$

12. $\begin{cases} \frac{1}{2}x + y = 3, \\ x + \frac{1}{2}y = 3. \end{cases}$

13. $\begin{cases} 2x + 3y = 6, \\ 3y = 3 - x. \end{cases}$

14. $\begin{cases} 4y - x = 4, \\ \frac{1}{2}x = 2y - 2. \end{cases}$

15. $\begin{cases} y = 3(x - 1), \\ 18 = 3(y + 2x). \end{cases}$

16. $\begin{cases} 3x = 4y, \\ 3x - 4y = 9. \end{cases}$

17. $\begin{cases} x - 2y = 4, \\ 2y + 6x = 3. \end{cases}$

18. $\begin{cases} 2y = 3(x - 2), \\ 9x = 6(y + 3). \end{cases}$

19. $\begin{cases} 2x + 4y = -8, \\ x - 3y = -4. \end{cases}$

20. $\begin{cases} 2x + 3y = 9, \\ 6y + 4x = 18. \end{cases}$

21. $\begin{cases} 3x + 2y = 12, \\ 2y - x = 12. \end{cases}$

22. During a certain month (July 1–31) one year the average daily maximum temperature for ten cities in the United States was as follows: 80°; 80°; 82°; 82°; 78°; 80°; 81°; 84°; 84°; 84°; 86°; 86°; 85°; 86°; 90°; 89°; 91°; 89°; 87°; 86°; 83°; 82°; 80°; 81°; 82°; 82°; 82°; 82°; 85°; 85°; 84°.

Draw the graph with each horizontal space representing 1 day, and each vertical space 1 degree of temperature.

23. On November 1 of each year from 1909–1913, the wholesale price of wheat per bushel was as follows: 1909, \$ 1.23½; 1910, \$.96; 1911, \$.99½; 1912, \$ 1.06; 1913, \$.98. Draw a graph showing the comparative prices for the five years, letting 4 horizontal spaces represent 1 year and each vertical space 2 ¢.

24. The cotton crop of Texas given in million bales for years 1907–1913 was as follows: 1907, 4.07; 1908, 2.31; 1909, 3.91; 1910, 2.65; 1911, 3.14; 1912, 4.27; 1913, 4.88.

Draw a graph showing the variation in the crop for these years, with 2 horizontal spaces representing 1 year and each vertical space ½ of a million bales of cotton.

25. The charge for sending parcels of merchandise weighing from 1 pound to 50 pounds not more than 50 miles by mail is 5¢ for 1 pound and 1¢ for each additional pound.

Draw a graph showing the charges on parcels weighing from 1 pound to 50 pounds, letting each horizontal space represent 2 pounds and each vertical space 2¢.

26. Letting two horizontal spaces represent 1 year and each vertical space 1 inch, construct two graphs showing the comparative heights of two standard types ("slender" and "heavy") of boys between the ages of 9 years and 15 years:

Age	9	10	11	12	13	14	15
Slender	53	$54\frac{1}{2}$	56	$58\frac{1}{2}$	60	63	66
Heavy	51	$52\frac{1}{2}$	$54\frac{1}{2}$	$56\frac{3}{4}$	$58\frac{3}{4}$	$60\frac{1}{2}$	$63\frac{3}{4}$

27. Draw two graphs showing the comparative chest girths in inches of boys of the two types mentioned in exercise 26:

Age	9	10	11	12	13	14	15
Slender	23	$23\frac{1}{2}$	$24\frac{1}{2}$	26	$26\frac{1}{2}$	27	$27\frac{1}{2}$
Heavy	25	26	$26\frac{1}{2}$	28	29	30	32

28. Draw a graph showing the amount of interest at 6% due on $1000 for different periods of time between 1 month and 2 years. (Use a convenient scale.)

29. At 7.00 A.M., Mr. Cox started for a town 18 miles distant, walking at the rate of 4 miles per hour. After walking for 2 hours he rested for a half hour. Draw a graph showing at what time he reached his destination.

30. Train No. 1 started from A at 9.00 A.M., traveling toward C, 120 miles away, at the rate of 50 miles per hour. At B, a station halfway between A and C, the train was detained 18 minutes, but it made no other stop. At 10.00 A.M., train No. 2 started from C, traveling toward A at the rate of 55 miles per hour and making no stops. Using any convenient scale, draw a graph showing where the trains met.

INVOLUTION AND EVOLUTION

154. Define **power**; **involution**; **root**; **evolution.**

INVOLUTION

155. The following *laws for involution,* which are the direct consequences of the corresponding laws for multiplication, are applicable in finding powers of monomial expressions:

Law of signs. — *All powers of a positive number are positive; even powers of a negative number are positive, and odd powers are negative.*

Thus, $2^2 = 4$; $(-2)^2 = 4$; and $(-2)^3 = -8$.

Law of exponents. — *The exponent of a power of a number is equal to the exponent of the number multiplied by the exponent of the power to which the number is to be raised.*

Thus, $(2^2)^3 = 2^{2\times 3} = 2^6 = 64$; also $(3^2)^2 = 3^{2\times 2} = 3^4 = 81$.

Distributive law. — *Any power of a product is equal to the product of its factors each raised to that power.*

Any power of the quotient of two numbers is equal to the quotient of the numbers each raised to that power.

Thus, $(2\cdot 3)^2 = (2\cdot 3)(2\cdot 3) = 2\cdot 2\cdot 3\cdot 3 = 2^2\cdot 3^2$; and $(\frac{2}{3})^2 = \frac{2}{3}\cdot\frac{2}{3} = \frac{2^2}{3^2}$.

EXERCISES

156. Raise to the power indicated:

1. $(x^2yz^3)^2$.	6. $(abc)^n$.	11. $(-1)^{2n}$.
2. $(ab^3c^3)^3$.	7. $(2\,x^2y^c)^5$.	12. $(-b^2)^{2n+1}$.
3. $(-3\,ab)^3$.	8. $(-2\,l^4m^5d^2)^3$.	13. $(-a^{2n}y^{3p}z^{4r})^5$.
4. $(2\,ax^2y)^4$.	9. $(-a^2x^ny^{n-1})^2$.	14. $(-a^{n-1}b^{n-2}c)^3$.
5. $(-6\,a^2x^3)^2$.	10. $(-x^3y^3z^{n-3})^3$.	15. $[-2(a-b)^2]^2$.

Raise to the power indicated:

16. $\left(\dfrac{5\,a^2}{12\,b^3}\right)^2$.
19. $\left(-\dfrac{2\,c}{3\,d^2}\right)^4$.
22. $\left(\dfrac{a^{n-1}b}{x^{2b}y^a}\right)^n$.

17. $\left(\dfrac{2\,x^2}{3\,y}\right)^3$.
20. $\left(-\dfrac{2\,a}{x^2y}\right)^5$.
23. $\left(\dfrac{a^{b-1}c^n}{x^{m+n-1}}\right)^n$.

18. $\left(\dfrac{a^n}{2\,b^{n-1}}\right)^3$.
21. $\left(-\dfrac{x^ny^n}{a^2}\right)^7$.
24. $\left(-\dfrac{x^3y^a}{c^nd}\right)^{2n}$.

Binomial Theorem

157. By actual multiplication, we have

Expansion of $(a+x)^n$	Binomial Coefficients
$(a+x)^0 = 1$	1
$(a+x)^1 = a + x$	1 1
$(a+x)^2 = a^2 + 2\,ax + x^2$	1 2 1
$(a+x)^3 = a^3 + 3\,a^2x + 3\,ax^2 + x^3$	1 3 3 1
$(a+x)^4 = a^4+4\,a^3x+6\,a^2x^2+4\,ax^3+x^4$	1 4 6 4 1
$(a+x)^5 = a^5+5\,a^4x+10\,a^3x^2+10\,a^2x^3+5\,ax^4+x^5$	1 5 10 10 5 1

The triangular array of coefficients is known as **Pascal's triangle.** Each number is the sum of the one directly above and the one to the left of that.

158. From the expansions given above, it will be observed that for any *positive integral* value of n in the **expansion of** $(a+x)^n$:

1. *The first term is* a^n, *the last term is* x^n, *and the number of terms is* $n+1$.

2. *The exponent of* a *decreases* 1 *in each succeeding term;* x *appears in the second term and its exponent increases* 1 *in each succeeding term.*

3. *The coefficient of any term may be found by multiplying the coefficient of the preceding term by the exponent of* a *in that term, and dividing this product by the number of the term.* the p

4. *All terms are positive, if both* a *and* x *are positive, and alternately positive and negative, if* x *is negative.*

Note that the coefficients of the latter half of the expansion are the same as those of the first half, written in reverse order.

EXERCISES

159. Write by inspection the expansion of:

1. $(x-a)^3$. 3. $(b+y)^5$. 5. $(r+s)^6$. 7. $(u-v)^6$.

2. $(x-a)^4$. 4. $(c-d)^5$. 6. $(r+s)^7$. 8. $(m-n)^7$.

9. Expand $(3a-b^2)^4$.

SOLUTION

$$(3a-b^2)^4 = (3a)^4 - 4(3a)^3(b^2) + 6(3a)^2(b^2)^2 - 4(3a)(b^2)^3 + (b^2)^4$$
$$= 81a^4 - 108a^3b^2 + 54a^2b^4 - 12ab^6 + b^8.$$

Test the result by giving the letters numerical values.

Expand, testing the results in exercises 10–15:

10. $(x+2)^3$.
11. $(a-3)^4$.
12. $(m-pn)^4$.
13. $(ax-by)^3$.
14. $(ax+2y)^3$.
15. $(2a+bc)^4$.
16. $(x^2-5y)^3$.
17. $(3c+d^2)^4$.
18. $(2ab-c)^3$.
19. $(x^3+2yz)^3$.
20. $(1-3y)^4$.
21. $(1+x^2y^2)^4$.
22. $(y^2-10)^3$.
23. $(1-2b)^5$.
24. $(2a-3c)^3$.
25. $(a^2x+4)^5$.
26. $(1-3a)^5$.
27. $(3a+bd)^6$.
28. $(\frac{1}{2}m+\frac{1}{3}n)^4$.
29. $(2x+\frac{1}{4}y)^5$.
30. $\left(\frac{a}{b}+\frac{b}{a}\right)^4$.
31. $\left(\frac{a}{b}-\frac{b}{a}\right)^7$.
32. $\left(y-\frac{1}{y}\right)^5$.
33. $(\frac{3}{4}-2x)^4$.
34. $\left(\frac{1}{3a}+3a\right)^5$.
35. $\left(2a^2-\frac{b^3}{2}\right)^5$.

36. Expand $(a+b-c)^3$.

SUGGESTION. $(a+b-c)^3=(\overline{a+b}-c)^3$, a binomial form. Expand this and then expand each binomial factor in the terms of the resulting expansion.

37. Expand $(r-s-t+v)^3$.

SUGGESTION. $(r-s-t+v)^3=(\overline{r-s}-\overline{t-v})^3$, a binomial form.

Expand:

38. $(a-b+c)^3$.
39. $(x+y+2)^3$.
40. $(b-c-d)^3$.
41. $(x+3y-2z)^3$.
42. $(a+b+c+d)^3$.
43. $(a+b-x-y)^3$.

160. The product of the successive integers from 1 to n, or from n to 1, inclusive, is called **factorial n**, written $\underline{|n}$, or $n\,!$.

$\underline{|5} = 1 \times 2 \times 3 \times 4 \times 5$, or $5 \times 4 \times 3 \times 2 \times 1$;
$\underline{|n} = 1 \cdot 2 \cdot 3 \cdots (n-2)(n-1)n$, or $n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1$.

161. Finding the *r*th term of the expansion of $(a+x)^n$.

From the expansions given in § 157 and the observations in § 158, it is evident that the following powers of $(a+x)$ may be written:

$$(a+x)^2 = a^2 + 2\,ax + \frac{2 \cdot 1}{1 \cdot 2}x^2.$$

$$(a+x)^3 = a^3 + 3\,a^2x + \frac{3 \cdot 2}{1 \cdot 2}ax^2 + \frac{3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3}x^3.$$

$$(a+x)^4 = a^4 + 4\,a^3x + \frac{4 \cdot 3}{1 \cdot 2}a^2x^2 + \frac{4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3}ax^3 + \frac{4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3 \cdot 4}x^4.$$

If the law of development revealed in the above is applied to the expansion of any power of any binomial, as the nth power of $(a+x)$, the result is

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{\underline{|2}}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{\underline{|3}}a^{n-3}x^3 \cdots.$$

This is known as the **binomial formula.**

From the binomial formula, it is evident that in *any term:*

1. *The exponent of* **x** *is 1 less than the number of the term.*

2. *The exponent of* **a** *is* **n** *minus the exponent of* **x**.

3. *The number of factors in the numerator and in the denominator of the coefficient is 1 less than the number of the term.*

Hence, the **formula for the *r*th term** of the expansion may be written:

$$\frac{n(n-1)(n-2)\cdots(n-r+2)}{\underline{|r-1}}a^{n-r+1}x^{r-1}.$$

Any term of the expansion of a power of a binomial may be obtained by substitution in this formula for the rth term.

In the expansion of a power of the difference of two numbers $(a-x)^n$, since the exponent of x in the rth term is $r-1$, the sign of the general term is $-$ if r is *even*, and $+$ if r is *odd*.

EXERCISES

162. 1. Find the 12th term of $(a-b)^{14}$.

SOLUTION

$$12\text{th term} = \frac{14\cdot 13\cdot 12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9\cdot 10\cdot 11}a^3(-b)^{11}$$

$$= -\frac{14\cdot 13\cdot 12}{1\cdot 2\cdot 3}a^3b^{11} = -364\,a^3b^{11}.$$

Or, since there are 15 terms, the coefficient of the 12th term, or the 4th term from the end, is equal to that of the 4th term from the beginning.

$$\therefore\ 12\text{th term} = -\frac{14\cdot 13\cdot 12}{1\cdot 2\cdot 3}a^3b^{11} = -364\,a^3b^{11}.$$

Without actually expanding, find the:

2. 4th term of $(a+2)^{10}$.
3. 8th term of $(x-y)^{11}$.
4. 5th term of $(x-2y)^{12}$.
5. 20th term of $(1+x)^{24}$.
6. 18th term of $(1-2x)^{20}$.
7. 13th term of $\left(a^2-\dfrac{1}{a^2}\right)^{14}$.
8. Find the middle term of $(a+3b)^6$.
9. Find the sixth term of $\left(x+\dfrac{1}{x}\right)^{10}$.
10. Find the middle term of $\left(\dfrac{x}{y}-\dfrac{y}{x}\right)^8$.
11. Find the two middle terms of $\left(\dfrac{a}{b}-\dfrac{b}{a}\right)^9$.
12. In the expansion of $(x^2+x)^{11}$, find the term containing x^{15}.

SOLUTION. — Since $(x^2+x)^{11} = \left[x^2\left(1+\dfrac{1}{x}\right)\right]^{11} = x^{22}\left(1+\dfrac{1}{x}\right)^{11}$, every term of the series expanded from $\left(1+\dfrac{1}{x}\right)^{11}$ will be multiplied by x^{22}.

Hence, the term sought is that which contains $\left(\dfrac{1}{x}\right)^7$, or $\dfrac{1^7}{x^7}$; that is, the 8th term, which is the same as the 5th term.

$$\therefore\ 8\text{th term} = x^{22}\frac{11\cdot 10\cdot 9\cdot 8}{1\cdot 2\cdot 3\cdot 4}\left(\frac{1}{x}\right)^7 = 330\,x^{15}.$$

13. Find the coefficient of a^9 in the expansion of $(a^3+a)^5$.
14. Which term contains $a^{18}b^9$ in the expansion of $(a-b)^{27}$?

EVOLUTION

163. Define and illustrate:

1. **Index of a root.**
2. **Odd and even root.**
3. **Square, cube, and fourth root.**
4. **Real and imaginary number.**

164. Since *evolution* is the *inverse of involution*, in general: *The nth root of* ***a*** *is a number of which the nth power is* ***a***.

165. *Every number has two square roots, one positive and the other negative.*

Thus, $\sqrt{25} = +5$ or -5, often written together ± 5 or ∓ 5.

166. Just as every number has two square roots, so every number has three cube roots, four fourth roots, etc.

The cube roots of 8, found later, are 2, $-1+\sqrt{-3}$, and $-1-\sqrt{-3}$.

The present discussion is concerned only with *real* roots.

167. A real root of a number, if it has the same sign as the number itself, is called a **principal root** of the number.

The principal square root of 25 is 5, not -5; the principal cube root of 8 is 2; of -8 is -2.

168. These *laws* for *evolution* follow from the corresponding laws for involution (§ 155):

Law of signs. — *An even root of a positive number may have either sign. An odd root of a number has the same sign as the number.*

Law of exponents. — *The exponent of any root of a number equals the exponent of the number divided by the index of the root.*

For the principal root, $\sqrt[3]{2^6} = 2^{6 \div 3} = 2^2 = 4$.

Distributive law. — *Any root of a product is equal to the product of that root of each of the factors.*

Any root of the quotient of two numbers is equal to the root of the dividend divided by the root of the divisor.

For principal roots, $\sqrt{25\,a^2} = \sqrt{25} \cdot \sqrt{a^2} = 5\,a$; and $\sqrt[4]{\dfrac{a^4}{b^4}} = \dfrac{\sqrt[4]{a^4}}{\sqrt[4]{b^4}} = \dfrac{a}{b}\cdot$

EXERCISES

169. Find the principal roots indicated:

1. $\sqrt{a^4b^2c^6}$.
2. $\sqrt[4]{x^8y^4z^8}$.
3. $\sqrt[3]{b^9c^6d^{12}}$.
4. $\sqrt[6]{l^6m^{12}n^{18}}$.
5. $\sqrt[5]{a^{5r}x^{10}y^{15}}$.
6. $\sqrt[3]{64\,x^3z^9}$.
7. $\sqrt{144\,a^4x^8}$.
8. $\sqrt[3]{-27\,a^6b^3}$.
9. $-\sqrt[5]{32\,a^{10}b^5}$.
10. $-\sqrt[3]{(-x^2y)^6}$.
11. $-\sqrt[8]{a^{16}b^{8n}c^{24}}$.
12. $-\sqrt[5]{-243\,y^{20}}$.
13. $\sqrt[5]{\dfrac{-1024\,a^5}{32\,x^5y^{10}}}$.
14. $\sqrt[3]{-729\,\dfrac{a^6}{b^3}}$.
15. $\sqrt[n]{\dfrac{b^{4n}c^nx^{3n}}{2^nx^{2n}y^{3n}}}$.
16. $\sqrt[2n]{\dfrac{a^{2n^2-2n}b^{4n}}{x^{4n}y^{2n}}}$.

Square Root of Polynomials

EXERCISES

170. 1. Find the process for extracting the square root of $a^2+2\,ab+b^2$.

PROCESS

$$
\begin{array}{lll}
 & a^2+2\,ab+b^2 & \underline{|\,a+b} \\
 & \underline{a^2\qquad\qquad} & \\
\text{Trial divisor, } 2\,a & \quad |\,2\,ab+b^2 & \\
\text{Complete divisor, } 2\,a+b & \quad |\,\underline{2\,ab+b^2} &
\end{array}
$$

EXPLANATION. — Since $a^2+2\,ab+b^2$ is the square of $(a+b)$, we know that the square root of $a^2+2\,ab+b^2$ is $a+b$.

Since the first term of the root is a, it may be found by taking the square root of a^2, the first term of the power. On subtracting a^2, there is a remainder of $2\,ab+b^2$.

The second term of the root is known to be b, and that may be found by dividing the first term of the remainder by twice the part of the root already found. This divisor is called a *trial* divisor.

Since $2\,ab+b^2$ is equal to $b(2\,a+b)$, the complete divisor which multiplied by b produces the remainder $2\,ab+b^2$ is $2\,a+b$; that is, the complete divisor is found by adding the second term of the root to twice the root already found.

On multiplying the complete divisor by the second term of the root and subtracting, there is no remainder; then, $a+b$ is the required root.

Since, in squaring $a+b+c$, $a+b$ may be represented by x, and the square of the number by $x^2+2\,xc+c^2$, the square root of a number whose root consists of *more than two terms* may be extracted in the same way as in exercise 1, *by considering the terms already found as one term.*

2. Find the square root of $x^4+4\,x^3-6\,x^2-20\,x+25$.

PROCESS

$$
\begin{array}{l|l}
 & x^4+4\,x^3-6\,x^2-20\,x+25 \;\underline{|\,x^2+2\,x-5} \\
 & \underline{x^4} \\
2\,x^2 & 4\,x^3-\;\;6\,x^2 \\
2\,x^2+2\,x & \underline{4\,x^3+\;\;4\,x^2} \\
2\,x^2+4\,x & -10\,x^2-20\,x+25 \\
2\,x^2+4\,x-5 & \underline{-10\,x^2-20\,x+25}
\end{array}
$$

Explanation. — Proceeding as in exercise 1, we find that the first two terms of the root are $x^2+2\,x$.

Considering $x^2+2\,x$ as the first term of the root, we find the next term of the root as we found the second term, by dividing the remainder by twice the part of the root already found. Hence, the trial divisor is $2\,x^2+4\,x$, and the next term of the root is -5. Annexing this, as before, to the trial divisor already found, we find that the complete divisor is $2\,x^2+4\,x-5$. Multiplying this by -5 and subtracting the product from $-10\,x^2-20\,x+25$, we have no remainder.

Hence, the square root of the number is $x^2+2\,x-5$.

Rule. — *Arrange the terms of the polynomial with reference to the consecutive powers of some letter.*

Extract the square root of the first term, write the result as the first term of the root, and subtract its square from the given polynomial.

Divide the first term of the remainder by twice the root already found, as a trial divisor, and the quotient will be the next term of the root. Write this result in the root, and annex it to the trial divisor to form a complete divisor.

Multiply the complete divisor by this term of the root, and subtract the product from the first remainder.

Continue in this manner until all the terms of the root are found.

Extract the square root of:

3. $16x^4 + 24x^2 + 9$.
4. $1 + 50a^3 + 625a^6$.
5. $9y^2 + 60yz + 100z^2$.
6. $4x^2 + 2xy + \frac{1}{4}y^2$.
7. $\frac{4}{9}d^6 - \frac{2}{3}d^3n^2 + \frac{1}{4}n^4$.
8. $(a+b)^2 - 4(a+b) + 4$.
9. $16 + 16x - 20x^2 - 12x^3 + 9x^4$.
10. $a^8 + 12a^4b^4 - 16a^2b^6 - 4a^6b^2 + 16b^8$.
11. $a^4 - 2a^2b + 2a^2c^2 - 2bc^2 + b^2 + c^4$.
12. $4a^2 - 12ab + 16ac + 9b^2 + 16c^2 - 24bc$.
13. $9x^2 + 25y^2 + 9z^2 - 30xy + 18xz - 30yz$.
14. $\dfrac{25}{4n^2} + 15 + 9n^2$.
15. $\dfrac{x^4}{64} + \dfrac{x^3}{8} - x + 1$.
16. $x^2 + 2x - 1 - \dfrac{2}{x} + \dfrac{1}{x^2}$.
17. $x^4 + x^3 + \dfrac{x}{5} + \dfrac{13x^2}{20} + \dfrac{1}{25}$.
18. $\dfrac{4m^6}{9} - \dfrac{4m^5}{3} + \dfrac{19m^4}{15} + \dfrac{3m^3}{5} - \dfrac{73m^2}{50} + \dfrac{3m}{10} + \dfrac{9}{16}$.
19. $r^8 - \frac{4}{5}r^7 + \frac{4}{25}r^6 + \frac{5}{3}r^5 - \frac{20}{3}r^4 + \frac{12}{5}r^3 + \frac{25}{36}r^2 - 5r + 9$.

Find the square root to four terms:

20. $1 - a$.
21. $a^2 + 1$.
22. $x^2 - 1$.
23. $4 - a$.
24. $y^2 + 3$.
25. $a^2 + 2b$.

Square Root of Arithmetical Numbers

171. Compare the number of digits in each number and its square root:

Number	Root	Number	Root	Number	Root
1	1	1′00	10	1′00′00	100
81	9	98′01	99	99′80′01	999

Principle. — *If a number is separated into periods of two digits each, beginning at units, its square root will have as many digits as the number has periods.*

The left-hand period may be incomplete, consisting of only one digit.

172. If the number of units expressed by the tens' digit is represented by t and the number of units expressed by the units' digit, by u, any number consisting of tens and units may be represented by $t + u$, and its square by $(t + u)^2$, or

$$t^2 + 2\,tu + u^2.$$

Since $25 = 20 + 5$, $25^2 = (20 + 5)^2 = 20^2 + 2\,(20 \times 5) + 5^2 = 625$.

EXERCISES

173. 1. Extract the square root of 5329.

FIRST PROCESS

$$\begin{array}{r|r|l}
 & 53'29 & 70 + 3 \\
t^2 = & 49\;00 & \\
\hline
2\,t = 140 & 4\;29 & \\
u = \quad 3 & & \\
\hline
2\,t + u = 143 & 4\;29 & \\
\end{array}$$

SECOND PROCESS

$$\begin{array}{r|r|l}
 & 53'29 & 73 \\
 & 49 & \\
\hline
140 & 4\;29 & \\
3 & & \\
\hline
143 & 4\;29 & \\
\end{array}$$

EXPLANATION. — Separating the number into periods of two digits each (§ 171), we find that the root is composed of two digits, tens and units. Since the largest square in 53 is 7, the tens of the root cannot be greater than 7 tens, or 70. Writing 7 tens in the root, squaring, and subtracting from 5329, we have a remainder of 429.

Since the square of a number composed of tens and units is equal to (*the square of the tens*) + (*twice the product of the tens and the units*) + (*the square of the units*), when the square of the tens has been subtracted, the remainder, 429, is twice the product of the tens and the units, plus the square of the units, or only a little more than twice the product of the tens and the units.

Therefore, 429 divided by twice the tens is approximately equal to the units. 2×7 tens, or 140, then, is a *trial*, or *partial*, *divisor*. On dividing 429 by the trial divisor, the units' figure is found to be 3.

Since twice the tens are to be multiplied by the units, and the units also are to be multiplied by the units to obtain the square of the units, in order to abridge the process the tens and units are first added, forming the *complete divisor* 143, and then multiplied by the units. Thus, $(140 + 3)$ multiplied by $3 = 429$.

Therefore, the square root of 5329 is 73.

In practice it is usual to place the figures of the same order in the same column, and to disregard the ciphers on the right of the products, as in the second process.

Since any number may be regarded as composed of tens and units, the foregoing processes have a general application.

Thus, $346 = 34$ tens $+ 6$ units; $2377 = 237$ tens $+ 7$ units.

2. Find the square root of 137,641.

Solution

		13′76′41	371
		9	
Trial divisor	$= 2 \times 30 = 60$	4 76	
Complete divisor	$= 60 + 7 = 67$	4 69	
Trial divisor	$= 2 \times 370 = 740$	7 41	
Complete divisor	$= 740 + 1 = 741$	7 41	

RULE. — *Separate the number into periods of two figures each, beginning at units.*

Find the greatest square in the left-hand period and write its root for the first figure of the required root.

Square this root, subtract the result from the left-hand period, and annex to the remainder the next period for a new dividend.

Double the root already found, with a cipher annexed, for a trial divisor, and by it divide the dividend. The quotient, or quotient diminished, will be the second figure of the root. Add to the trial divisor the figure last found, multiply this complete divisor by the figure of the root last found, subtract the product from the dividend, and to the remainder annex the next period for the next dividend.

Proceed in this manner until all the periods have been used. The result will be the square root sought.

1. When the number is not a perfect square, annex periods of decimal ciphers and continue the process.
2. Decimals are pointed off from the decimal point toward the right.
3. The square root of a common fraction may be found by extracting the square root of both numerator and denominator separately or by reducing the fraction to a decimal and then extracting the root.

Extract the square root of:

3. 5776.	6. 86,436.	9. 4.5369.	12. 11.0224.
4. 9604.	7. 8.0089.	10. 864,900.	13. .633616.
5. 6241.	8. 655.36.	11. 576,081.	14. .994009.

Find the square root of:

15. $\frac{289}{576}$.	17. $\frac{625}{841}$.	19. $\frac{529}{900}$.	21. $\frac{4096}{9801}$.
16. $\frac{196}{729}$.	18. $\frac{169}{361}$.	20. $\frac{961}{1296}$.	22. $\frac{1089}{9025}$.

Extract the square root to four decimal places:

23. 8.	25. $\frac{5}{16}$.	27. 2.5.	29. .7854.
24. 7.	26. $\frac{4}{15}$.	28. 3.6.	30. .41265.

31. Find, to the nearest tenth of a rod, the side of a square garden that contains 2 acres.

32. How many rods of fence are required to inclose a square field whose area is 10 acres?

33. The legs of a right triangle are 12 feet and 15 feet. Find, to the nearest foot, the length of the hypotenuse.

SUGGESTION. — Since the square of the hypotenuse of a right triangle equals the sum of the squares of the other two sides, if x represents the hypotenuse, $x^2 = 12^2 + 15^2$.

34. What is the length, to the nearest tenth of a foot, of the diagonal of a 5-foot square?

35. Two vessels sailed from the same point, one north at the rate of 15 knots an hour, the other east at the rate of 20 knots an hour. How far apart were they after 6 hours?

36. The length of the hypotenuse of a right triangle is 18 inches and the length of one side is 14 inches. Find, to the nearest inch, the length of the other side.

37. A rectangular field is 40 rods long and 25 rods wide. Find, to the nearest tenth of a rod, the length of a path extending diagonally across the field.

38. A 30-foot ladder leans against a wall, with the foot $5\frac{1}{2}$ feet from the wall. Find, to the nearest hundredth of a foot, the height of the top of the ladder.

EXPONENTS AND RADICALS

THEORY OF EXPONENTS

174. Thus far the exponents used have been *positive integers* only, and consequently the **laws of exponents** have been obtained in the following restricted forms:

I. $a^m \times a^n = a^{m+n}$ when m and n are positive integers.

II. $a^m \div a^n = a^{m-n}$ when m and n are positive integers and m is greater than n.

III. $(a^m)^n = a^{mn}$ when m and n are positive integers.

IV. $\sqrt[n]{a^m} = a^{m \div n}$ when m and n are positive integers and m is a multiple of n.

V. $(ab)^n = a^n b^n$ when n is a positive integer.

These laws may be *proved* as follows:

I. Let m and n be any positive integers, and let a be any number.

By notation, $a^m = a \cdot a \cdot a \cdots$ to m factors,

and $a^n = a \cdot a \cdot a \cdots$ to n factors;

$\therefore\ a^m \cdot a^n = (a \cdot a \cdot a \cdots$ to m factors$)(a \cdot a \cdot a \cdots$ to n factors$)$

by assoc. law, $= a \cdot a \cdot a \cdots$ to $(m+n)$ factors

by notation, $= a^{m+n}$.

II. Let m and n be positive integers, m being greater than n, and let a be any number.

By notation, $a^m = a \cdot a \cdot a \cdots$ to m factors,

and $a^n = a \cdot a \cdot a \cdots$ to n factors;

$$\therefore\ \frac{a^m}{a^n} = \frac{a \cdot a \cdot a \cdots \text{ to } m \text{ factors}}{a \cdot a \cdot a \cdots \text{ to } n \text{ factors}}.$$

Remove equal factors from dividend and divisor.

Then, $a^m \div a^n = a \cdot a \cdot a \cdots$ to $(m-n)$ factors

by notation, $= a^{m-n}$.

III. Let m and n be positive integers, and let a be any number.

By notation, $(a^m)^n = a^m \cdot a^m \cdot a^m \cdots$ to n factors

by law I, $= a^{m+m+m+\cdots \text{ to } n \text{ terms}}$

by notation, $= a^{mn}$.

IV. Let r and s be positive integers, and let a be any number.

By law III, $(a^r)^s = a^{rs}$. (1)

Indicating the sth root of each member, Ax. 7, we have

$$\sqrt[s]{(a^r)^s} = \sqrt[s]{a^{rs}}. \quad (2)$$

But by § 164, $\sqrt[s]{(a^r)^s} = a^r$; $\therefore$ from (2), Ax. 5, $\sqrt[s]{a^{rs}} = a^r$.

That is, $\sqrt[s]{a^{rs}} = a^{rs \div s} = a^r$.

In general, when m and n are positive integers, m being a multiple of n, and a any number,

$$\sqrt[n]{a^m} = a^{m \div n}.$$

V. Let n be a positive integer, and let a and b be any numbers.

By notation, $(ab)^n = ab \cdot ab \cdot ab \cdots$ to n factors

by assoc. law, $= (a \cdot a \cdot a \cdots$ to n factors$)(b \cdot b \cdot b \cdots$ to n factors$)$

by notation, $= a^n b^n$.

If all restrictions are removed from m and n, we may then have expressions like a^0, a^{-2}, and $a^{\frac{3}{5}}$. But such expressions are as yet unintelligible, because -2 and $\frac{3}{5}$ cannot show how many times a number is used as a factor, and the meaning of a^0 has not yet been explained.

Since, however, these forms may occur in algebraic processes, it is important to discover meanings for them that will allow their use in accordance with the laws already established, for otherwise great complexity and confusion would arise in the processes involving them.

Assuming that the law of exponents for multiplication,

$$a^m \times a^n = a^{m+n},$$

is true for *all* values of m and n, the meanings of *zero, negative,* and *fractional* exponents may be discovered readily by substituting these different kinds of exponents for m and n or both, and observing to what conclusions we are led.

175. Meaning of a zero exponent.

We have agreed that any new kind of exponent shall have its meaning determined in harmony with the law of exponents for multiplication, expressed by the formula,

$$a^m \times a^n = a^{m+n}.$$

If $n = 0$, $\qquad a^m \times a^0 = a^{m+0}$, or a^m.

Dividing by a^m, Ax. 4, $\qquad a^0 = \dfrac{a^m}{a^m} = 1.$ That is,

Any number (not zero) with a zero exponent is equal to 1.

176. Meaning of a negative exponent.

Since, § 174, $a^m \times a^n = a^{m+n}$ is to hold true for all values of m and n, if $m = -n$,

$$a^{-n} \times a^n = a^{-n+n} = a^0.$$

But, § 175, $\qquad a^0 = 1.$

Hence, Ax. 5, $\qquad a^{-n} \times a^n = 1.$

Dividing by a^n, Ax. 4, $\qquad a^{-n} = \dfrac{1}{a^n}\cdot$ That is,

Any number with a negative exponent is equal to the reciprocal of the same number with a numerically equal positive exponent.

177. By the definition of negative exponent just given,

$$a^{-m} = \frac{1}{a^m} \text{ and } b^{-n} = \frac{1}{b^n}\cdot$$

Therefore, $\qquad \dfrac{a^{-m}}{b^{-n}} = \dfrac{\dfrac{1}{a^m}}{\dfrac{1}{b^n}} = \dfrac{1}{a^m} \times \dfrac{b^n}{1} = \dfrac{b^n}{a^m}\cdot$ Hence,

Any factor may be transferred from one term of a fraction to the other without changing the value of the fraction, provided the sign of the exponent is changed.

178. Meaning of a fractional exponent.

Since (§ 174) the first law of exponents is to hold true for all exponents, $a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^1 = a;$

that is, $a^{\frac{1}{2}}$ is one of the two equal factors of a, or is a square root of a.

Again, $$a^{\frac{3}{2}} \times a^{\frac{3}{2}} = a^{\frac{3}{2}+\frac{3}{2}} = a^3;$$

that is, $a^{\frac{3}{2}}$ is a square root of the cube of a.

Or, similarly, $a^{\frac{1}{2}} \times a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}+\frac{1}{2}} = a^{\frac{3}{2}};$

that is, $a^{\frac{3}{2}}$ is the cube of a square root of a.

In general, confining the discussion to principal roots, let p and q be any two positive integers. By the first law of exponents, § 174, $a^{\frac{p}{q}} \cdot a^{\frac{p}{q}} \cdots$ to q factors $= a^{\frac{p}{q}+\frac{p}{q}+\cdots \text{ to } q \text{ terms}} = a^p.$

Therefore $a^{\frac{p}{q}}$, one of the q equal factors of a^p, is a qth root of the pth power of a.

Similarly, $a^{\frac{1}{q}}$ is a qth root of a.

Also, since $a^{\frac{1}{q}} \cdot a^{\frac{1}{q}} \cdots$ to p factors $= a^{\frac{1}{q}+\frac{1}{q}+\cdots \text{ to } p \text{ terms}} = a^{\frac{p}{q}}$, $a^{\frac{p}{q}}$ is the pth power of a qth root of a.

The numerator of a fractional exponent with positive integral terms indicates a power and the denominator, a root.

The fraction as a whole indicates a root of a power or a power of a root.

The fractional exponent with the meaning just given comes directly from § 174, law IV, when the restriction that m *is a multiple of* n is removed; thus, $$\sqrt[n]{a^m} = a^{m \div n} = a^{\frac{m}{n}}.$$

179. Any fractional exponent that does not itself involve a root sign may be reduced to one of the forms $\frac{p}{q}$ or $-\frac{p}{q}$.

Thus, $$8^{\frac{-\frac{8}{3}}{2}} = 8^{-\frac{4}{3}}.$$

By §§ 176, 168, dis. law, $$a^{-\frac{p}{q}} = \frac{1}{a^{\frac{p}{q}}} = \frac{1^{\frac{p}{q}}}{a^{\frac{p}{q}}} = \left(\frac{1}{a}\right)^{\frac{p}{q}}.$$

EXERCISES

180. Find a simple value for:

1. 3^{-2}.
2. 4^0.
3. $(-4)^{-3}$.
4. $(-8)^0$.
5. $\dfrac{3}{5^{-2}}$.
6. $\dfrac{3^{-2}}{4}$.
7. $(-\frac{1}{2})^{-3}$.
8. $(x^3y^{2n}z^m)^0$.

9. Find the value of $2^3 - 3 \cdot 2^2 + 5 \cdot 2^1 - 7 \cdot 2^0 + 4 \cdot 2^{-1} - 2^{-2}$.

10. Find the value of $x^2 - 3x^1 + 4x^0 + x^{-1} - 5x^{-2} + x^{-3}$ when $x = \frac{1}{2}$; when $x = -\frac{1}{2}$; when $x = 1$.

11. Which is the greater, $(-\frac{1}{3})^{-3}$ or $(\frac{1}{3})^3$? $(-\frac{1}{3})^{-4}$ or $(\frac{1}{3})^4$?

Write with negative exponents:

12. $1 \div 5$.
13. $1 \div a^2$.
14. $1 \div 2^n$.
15. $a \div x^3$.
16. $m \div bn^2$.
17. $c \div a^2x^3$.
18. $c^2d \div a^2b^3$.
19. $am^3 \div bx^n$.

Write with positive exponents:

20. $3y^{-2}$.
21. $2ax^{-1}$.
22. $5x^{-1}z^2$.
23. $3a^{-1}b^{-2}$.
24. $xy^{-1}z^{-4}$.
25. $a^{-1}b^2c^{-3}$.
26. $2b^2c^{-2}d$.
27. $3x^{-4}y^{-1}z^2$.
28. a^nb^{-3n}.
29. $5a^2b^{-4}c^5$.
30. $3x^{-a}y^{-b}z^{-2c}$.
31. $5l^{2r}m^{-a}n^{-2b}$.
32. $4x^3 - 2x^2 + 5x^1 - 6x^0 + 3x^{-1} - 5x^{-2}$.
33. $2a^3 - 12a^2 - 16a + 3a^0 + 2a^{-1} - 7a^{-2}$.
34. $a^3b^{-3} - a^2b^{-2} + ab^{-1} - 1 + a^{-1}b - a^{-2}b^2 + a^{-3}b^3$.

Write without a denominator:

35. $\dfrac{c^2d^2}{a^2b^2}$.
36. $\dfrac{a^2m^3}{b^3n^2}$.
37. $\dfrac{b}{x^{-6}}$.
38. $\dfrac{1}{m^{-3}n}$.
39. $\dfrac{3a^2c^{-2}}{x^2}$.
40. $\dfrac{2a^2x^{-3}}{b^{-4}y}$.
41. $\dfrac{4}{x^{-4}}$.
42. $\dfrac{x^{-1}}{y}$.
43. $\left(\dfrac{3}{m}\right)^3$.
44. $\left(\dfrac{a}{b^2}\right)^3$.
45. $\dfrac{a^{-3}}{b^{-3}}$.
46. $\dfrac{1}{(ab)^3}$.

47. Find the value of $27^{\frac{2}{3}}$.

SOLUTION. — By § 178, $27^{\frac{2}{3}} = (\sqrt[3]{27})^2 = 3^2 = 9$.

Simplify, taking only principal roots:

48. $4^{\frac{1}{2}}$. **51.** $16^{\frac{1}{4}}$. **54.** $8^{-\frac{1}{3}}$. **57.** $81^{\frac{3}{4}}$.

49. $4^{\frac{3}{2}}$. **52.** $16^{\frac{3}{4}}$. **55.** $16^{-\frac{3}{4}}$. **58.** $64^{-\frac{2}{3}}$.

50. $9^{-\frac{1}{2}}$. **53.** $25^{\frac{3}{2}}$. **56.** $(-8)^{\frac{1}{3}}$. **59.** $(-32)^{-\frac{3}{5}}$.

60. Which is the greater, $64^{-\frac{2}{3}}$ or $(\frac{1}{64})^{\frac{3}{2}}$? $64^{\frac{2}{3}}$ or $(\frac{1}{64})^{-\frac{3}{2}}$?

61. Find the value of $x^{-\frac{4}{3}} - 4x^{-\frac{2}{3}} + 4$ when $x = -\frac{1}{125}$.

62. Express $\sqrt[3]{x^{-1}y^{-4}z^2}$ with positive fractional exponents.

SOLUTION. $\sqrt[3]{x^{-1}y^{-4}z^2} = x^{-\frac{1}{3}}y^{-\frac{4}{3}}z^{\frac{2}{3}} = \dfrac{z^{\frac{2}{3}}}{x^{\frac{1}{3}}y^{\frac{4}{3}}}$.

Express with positive fractional exponents:

63. $\sqrt{cd}$. **66.** $(\sqrt{xy})^3$. **69.** $\sqrt{x^3y^{-1}z^{-3}}$.

64. $\sqrt[3]{ab}$. **67.** $(\sqrt[3]{mn})^4$. **70.** $3\sqrt[3]{a^{-2}b^xc^{-1}}$.

65. $\sqrt{x^3y^5}$. **68.** $(\sqrt[5]{ab})^3$. **71.** $2\sqrt[3]{(a+b)^2}$.

Express roots with radical signs and powers with positive exponents:

72. $x^{\frac{1}{2}}$. **75.** $a^{\frac{1}{2}}b^{\frac{3}{2}}$. **78.** $m^{\frac{3}{5}}n^{\frac{4}{5}}$. **81.** $a^{\frac{1}{3}} \div b^{\frac{1}{3}}$.

73. $a^{\frac{3}{2}}$. **76.** $x^{\frac{2}{3}}y^{\frac{1}{3}}$. **79.** $a^{\frac{2}{3}}b^{-\frac{4}{3}}$. **82.** $c^{\frac{1}{4}} \div d^{\frac{5}{4}}$.

74. $y^{\frac{3}{4}}$. **77.** $c^{\frac{1}{4}}d^{-\frac{3}{4}}$. **80.** $a^{\frac{1}{5}}b^{\frac{2}{5}}c^{-\frac{3}{5}}$. **83.** $x^{\frac{3}{5}} \div y^{-\frac{4}{5}}$.

84. Simplify $\sqrt[3]{x^2} + x^{\frac{1}{3}} + 8^{\frac{2}{3}} + 3x^{\frac{2}{3}} - 5\sqrt[3]{x} - \sqrt[3]{27^2}$.

85. Simplify $4\sqrt[5]{x} + 5x^0 - 3x^{-\frac{1}{5}} + 2\sqrt[5]{x^{-1}} - 8^{\frac{2}{3}} - 2x^{\frac{1}{5}}$.

181. Operations involving positive, negative, zero, and fractional exponents.

Since zero, negative, and fractional exponents have been defined in conformity with the law of exponents for multiplication, this law holds true for all exponents so far encountered.

For the proofs of the generality of the other laws of exponents, see the author's *Academic Algebra.*

EXERCISES

182. Multiply:

1. a by a^{-4}.	**4.** $a^{\frac{1}{2}}b^{\frac{1}{3}}$ by $a^{\frac{1}{2}}b^{\frac{2}{3}}$.	**7.** n^{-2} by $an^{\frac{5}{2}}$.
2. a^2 by a^0.	**5.** $m^{\frac{2}{5}}n$ by $m^{\frac{3}{5}}n^{-1}$.	**8.** a^{m-n} by a^{n-p}.
3. $x^{\frac{1}{3}}$ by $x^{\frac{2}{3}}$.	**6.** $a^{\frac{1}{3}}b^{\frac{1}{4}}$ by $a^{-\frac{4}{3}}b^{\frac{3}{4}}$.	**9.** $a^{\frac{m+n}{2}}$ by $a^{\frac{m-n}{2}}$.

10. Multiply $x^{\frac{4}{5}}y^{-\frac{1}{5}} + x^{\frac{3}{5}} + x^{\frac{2}{5}}y^{\frac{1}{5}} + x^{\frac{1}{5}}y^{\frac{2}{5}} + y^{\frac{3}{5}}$ by $x^{\frac{1}{5}}y^{\frac{1}{5}}$.

11. Multiply $y^n + x^{-1}y^{n+1} + x^{-2}y^{n+2} + x^{-3}y^{n+3}$ by x^ny^{-n}.

12. Expand $(a^{\frac{1}{5}}b^{-\frac{1}{2}} + 1 + a^{-\frac{1}{5}}b^{\frac{1}{2}})(a^{\frac{1}{5}}b^{-\frac{1}{2}} - 1 + a^{-\frac{1}{5}}b^{\frac{1}{2}})$.

SOLUTION

$$\begin{array}{lllll}
a^{\frac{1}{5}}b^{-\frac{1}{2}} & + \quad 1 & + a^{-\frac{1}{5}}b^{\frac{1}{2}} & & \\
a^{\frac{1}{5}}b^{-\frac{1}{2}} & - \quad 1 & + a^{-\frac{1}{5}}b^{\frac{1}{2}} & & \\
\hline
a^{\frac{2}{5}}b^{-1} & + a^{\frac{1}{5}}b^{-\frac{1}{2}} & + a^0b^0 & & \\
& - a^{\frac{1}{5}}b^{-\frac{1}{2}} & - \quad 1 & - a^{-\frac{1}{5}}b^{\frac{1}{2}} & \\
& & + a^0b^0 & + a^{-\frac{1}{5}}b^{\frac{1}{2}} & + a^{-\frac{2}{5}}b \\
\hline
a^{\frac{2}{5}}b^{-1} & & + \quad 1 & & + a^{-\frac{2}{5}}b
\end{array}$$

Expand:

13. $(a^{\frac{1}{2}} + b^{\frac{1}{2}})(a^{\frac{1}{2}} - b^{\frac{1}{2}})$.	**16.** $(x^{\frac{2}{5}} - x^{\frac{1}{5}}y^{-\frac{1}{5}} + y^{-\frac{2}{5}})(x^{\frac{1}{5}} + y^{-\frac{1}{5}})$.
14. $(x^{\frac{2}{3}} + y^{\frac{2}{3}})(x^{\frac{2}{3}} - y^{\frac{2}{3}})$.	**17.** $(1 - x + x^2)(x^{-3} + x^{-2} + x^{-1})$.
15. $(x^{\frac{2}{5}} - 4)(x^{\frac{2}{5}} + 5)$.	**18.** $(a^{-1} + b^{-\frac{1}{2}} + c^{\frac{1}{2}})(a^{-1} + b^{-\frac{1}{2}} + 2c^{\frac{1}{2}})$.

Divide:

19. n^3 by n^4.
20. n^2 by n^0.
21. n^4 by n^{-4}.
22. $r^{\frac{3}{2}}$ by $r^{-\frac{5}{2}}$.
23. s^{-2} by $s^{-\frac{7}{2}}$.
24. x^{1-n} by x^{n-1}.
25. $x^4 + x^2y^2 + y^4$ by x^2y^2.
26. $a^{-4} - a^{-2}b + b^2$ by $a^{-2}b$.
27. $x^4 - 2\,ax^3 + a^3x - a^4$ by a^4x^4.

28. Divide $b^{-1} + 3\,a^{-\frac{1}{2}} - 10\,a^{-1}b$ by $a^{\frac{1}{2}}b^{-1} - 2$.

SOLUTION

$$
\begin{array}{l|l}
b^{-1} + 3\,a^{-\frac{1}{2}} - 10\,a^{-1}b & a^{\frac{1}{2}}b^{-1} - 2 \\ \cline{2-2}
\underline{b^{-1} - 2\,a^{-\frac{1}{2}}} & a^{-\frac{1}{2}} + 5\,a^{-1}b \\
\qquad\quad 5\,a^{-\frac{1}{2}} - 10\,a^{-1}b & \\
\qquad\quad \underline{5\,a^{-\frac{1}{2}} - 10\,a^{-1}b} &
\end{array}
$$

Divide:

29. $a - b$ by $a^{\frac{1}{3}} - b^{\frac{1}{3}}$.
30. $a + b$ by $a^{\frac{1}{5}} + b^{\frac{1}{5}}$.
31. $a^2 + b^2$ by $a^{\frac{2}{3}} + b^{\frac{2}{3}}$.
32. $x - 1$ by $x^{\frac{2}{3}} + x^{\frac{1}{3}} + 1$.
33. $x^{\frac{4}{7}} - 2 + x^{-\frac{4}{7}}$ by $x^{\frac{2}{7}} - x^{-\frac{2}{7}}$.
34. $3 - 4\,x^{-1} + x^{-2}$ by $x^{-1} - 3$.

Simplify the following:

35. $(a^{\frac{1}{2}})^2$.
36. $(a^{-\frac{1}{3}})^6$.
37. $(a^{-4})^2$.
38. $(-a^{\frac{1}{2}})^3$.
39. $(-a^2)^4$.
40. $(-a^{\frac{4}{5}})^{-1}$.
41. $\sqrt[4]{a^{-\frac{8}{3}}}$.
42. $\sqrt{x^{\frac{4}{3}}y^{-3}}$.
43. $\sqrt[3]{a^{-\frac{1}{2}}b^{-3}}$.
44. $(-\frac{1}{32}\,a^{10})^{-\frac{3}{5}}$.
45. $(\frac{1}{36}a^{-\frac{2}{3}}b^{\frac{1}{3}})^{-\frac{3}{2}}$.
46. $(\frac{1}{9}\,m^{-1}n^{-\frac{1}{2}})^{\frac{1}{2}}$.

Expand by the binomial formula:

47. $(a^{\frac{1}{2}} - b^{\frac{1}{2}})^2$.
48. $(a^{\frac{1}{2}} + b^{\frac{1}{3}})^3$.
49. $(a^{-1} - b^{\frac{2}{3}})^3$.
50. $(x^{-\frac{1}{2}} - y^{\frac{1}{2}})^4$.
51. $(a^{-\frac{1}{2}} + \frac{1}{2})^3$.
52. $(1 - x^{\frac{3}{2}})^4$.

Extract the square root of:

53. $b + 4\,b^{\frac{1}{2}}c^{\frac{1}{2}} + 4\,c - 2\,b^{\frac{1}{2}} - 4\,c^{\frac{1}{2}} + 1$.
54. $x^2 + 2\,x^{\frac{3}{2}} + 3\,x + 4\,x^{\frac{1}{2}} + 3 + 2\,x^{-\frac{1}{2}} + x^{-1}$.
55. $x^2 + y + 4\,z^{-2} - 2\,xy^{\frac{1}{2}} + 4\,xz^{-1} - 4\,y^{\frac{1}{2}}z^{-1}$.
56. $a + 4\,b^{\frac{2}{3}} + 9\,c^{\frac{1}{2}} - 4\,a^{\frac{1}{2}}b^{\frac{1}{3}} + 6\,a^{\frac{1}{2}}c^{\frac{1}{4}} - 12\,b^{\frac{1}{3}}c^{\frac{1}{4}}$.

57. Factor $4x^{-2}-9y^{-2}$, and express the result with positive exponents.

SOLUTION

$$4x^{-2}-9y^{-2}=(2x^{-1}+3y^{-1})(2x^{-1}-3y^{-1})=\left(\frac{2}{x}+\frac{3}{y}\right)\left(\frac{2}{x}-\frac{3}{y}\right).$$

Factor, expressing results with positive exponents:

58. $a^{-2}-b^{-2}$.

59. $9-x^{-2}$.

60. $16-a^{-4}$.

61. x^3-x^{-3}.

62. $27-b^{-3}$.

63. $b^{-3}+y^{-3}$.

64. a^2+2+a^{-2}.

65. $b^4-8+16b^{-4}$.

66. $12-x^{-1}-x^{-2}$.

Solve for values of x corresponding to principal roots:

67. $x^{\frac{1}{2}}=5$.

68. $x^{\frac{2}{3}}=9$.

69. $x^{\frac{3}{5}}=8$.

70. $x^{\frac{4}{3}}=16$.

71. $x^{-\frac{1}{2}}=12$.

72. $\frac{1}{4}x^{\frac{2}{3}}=25$.

73. $2x^{-\frac{3}{2}}=\frac{1}{32}$.

74. $\frac{1}{2}x^{\frac{3}{2}}=108$.

75. $x^{\frac{5}{2}}=243$.

76. $x^{\frac{3}{4}}-a^6=0$.

77. $x^{\frac{6}{5}}-64=0$.

78. $x^{-\frac{3}{2}}-27=0$.

Simplify, expressing results with positive exponents:

79. $\left(\frac{2^{-2}}{2^{-3}}\right)^2$.

80. $\left(\frac{2^{\frac{1}{3}}}{4^{\frac{1}{4}}}\right)^{-12}$.

81. $\left(\frac{\sqrt{3^{\frac{1}{3}}}}{\sqrt[3]{2^{-2}}}\right)^{-6}$.

82. $\left(\frac{9^{-3}}{x^{-4}y^{-2}}\right)^{-\frac{1}{2}}$.

83. $\left(\frac{27^{-\frac{1}{3}}}{a}\right)^{-1}$.

84. $\left(\frac{a^4b^{-3}}{a^3b^{-2}}\right)^2$.

85. $\left(\frac{16\,m^{-\frac{2}{3}}}{r^{-4}}\right)^{-\frac{3}{4}}$.

86. $\left(\frac{m^{-2}n^{-\frac{2}{3}}}{x^{\frac{1}{6}}}\right)^{-3}$.

87. $\left(\frac{x^{-3}\div x^{-5}}{4x^{-4}}\right)^{-\frac{1}{2}}$.

88. $\sqrt[3]{\frac{9^r\times 3^{2+r}}{27^r}}$.

89. $\frac{3a^{\frac{1}{4}}\times 4a^{-1}}{6\sqrt[4]{a^5}}$.

90. $\frac{6x^{-2}\div 3x^{-\frac{1}{2}}}{2x^{-3}\times\sqrt{x}}$.

91. $\frac{2^{-1}\times 2^{-2}}{4^{-2}\times 4^{-3}}$.

92. $\frac{32^{\frac{3}{5}}+125^{\frac{2}{3}}}{81^{\frac{3}{4}}+216^{\frac{1}{3}}}$.

93. $\frac{\sqrt[3]{a^2}\times\sqrt{b^3}}{\sqrt[4]{b^6}\times\sqrt[6]{a^{-2}}}$.

94. $\frac{x^2-y^2}{(x^{\frac{1}{2}}-y^{\frac{1}{2}})(x^{\frac{1}{2}}+y^{\frac{1}{2}})}$.

95. $\frac{a^{-\frac{1}{2}}\times a^{-\frac{5}{2}}\times a^3}{(a+b)^{-2}}$.

96. $\frac{x^2y^{\frac{3}{5}}\times\sqrt[3]{x^{-5}}\times x^{\frac{2}{3}}}{\sqrt[5]{y^{-2}}\times xy}$.

RADICALS

183. Define and illustrate:

1. **Radical.**
2. **Radicand.**
3. **Radical expression.**
4. **Rational number.**
5. **Rational factor.**
6. **Irrational number.**
7. **Rational expression.**
8. **Irrational expression.**
9. **Surd.**
10. **Order of a surd.**
11. **Quadratic surd.**
12. **Cubic surd.**
13. **Biquadratic surd.**
14. **Mixed surd.**
15. **Entire surd.**
16. **Similar radicals.**

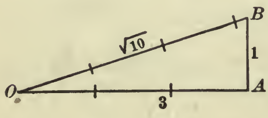
184. In the discussion and treatment of radicals only *principal roots* will be considered.

In the following pages it will be assumed that irrational numbers obey the same laws as rational numbers. For proofs of the generality of these laws, the reader is referred to the author's *Advanced Algebra.*

185. Graphical representation of a radical of the second order.

Since the hypotenuse of a right triangle is equal to the *square root of the sum of the squares of the other two sides,* a radical of the second order may be represented graphically by the *hypotenuse* of a right triangle whose other two sides are such that the sum of their squares is equal to the radicand.

Thus, to represent $\sqrt{10}$ graphically, since it may be observed that $10 = 3^2 + 1^2$, draw OA 3 units in length, then draw AB 1 unit in length in a direction perpendicular to OA. Draw OB, completing the right triangle OAB. Then, the length of OB represents $\sqrt{10}$ in its relation to the unit length.



Notice that $\sqrt{10}$ can be represented graphically by a line of *exact* length, though it cannot be represented exactly by decimal figures, for $\sqrt{10} = 3.162 \cdots$, an endless decimal.

EXERCISES

186. Represent graphically:

1. $\sqrt{18}$.
2. $\sqrt{20}$.
3. $\sqrt{29}$.
4. $\sqrt{32}$.
5. $\sqrt{41}$.
6. $\sqrt{53}$.
7. $\sqrt{\frac{10}{9}}$.
8. $\sqrt{\frac{73}{144}}$.

Reduction of Radicals

187. A radical is in its **simplest form** when the index of the root is as small as possible, and when the radicand is integral and contains no factor that is a perfect power whose exponent corresponds with the index of the root.

$\sqrt{7}$ is in its simplest form; but $\sqrt{\frac{3}{4}}$ is not in its simplest form, because $\frac{3}{4}$ is not integral in form; $\sqrt{8}$ is not in its simplest form, because the square root of 4, a factor of 8, may be extracted; $\sqrt[6]{25}$, or $25^{\frac{1}{6}}$, is not in its simplest form, because $25^{\frac{1}{6}} = (5^2)^{\frac{1}{6}} = 5^{\frac{2}{6}} = 5^{\frac{1}{3}}$, or $\sqrt[3]{5}$.

188. Reduction of a radical to its simplest form.

EXERCISES

1. Express $2\sqrt[3]{-384}$ in its simplest form.

SOLUTION. $2\sqrt[3]{-384} = 2\sqrt[3]{-64 \times 6} = 2\sqrt[3]{-64} \times \sqrt[3]{6} = -8\sqrt[3]{6}$.

RULE. — *Separate the radical into two factors one of which is its highest rational factor. Extract the required root of the rational factor, multiply the result by the coefficient, if any, of the given radical, and place the product as the coefficient of the irrational factor.*

Simplify:

2. $\sqrt{20}$.
3. $\sqrt[3]{24}$.
4. $\sqrt{32}$.
5. $\sqrt{50}$.
6. $\sqrt[3]{108}$.
7. $\sqrt{192}$.
8. $\sqrt[4]{162}$.
9. $\sqrt[3]{-16}$.
10. $\sqrt{72a^3}$.
11. $\sqrt{64b}$.
12. $\sqrt{128c^3}$.
13. $\sqrt[3]{-40x^5}$.
14. $\sqrt[5]{-486}$.
15. $\sqrt[5]{640y^5}$.
16. $\sqrt{98a^3b^4c}$.
17. $\sqrt[3]{250x^4y^7}$.
18. $(245a^6y^{-4})^{\frac{1}{2}}$.
19. $(x^3 - 2x^2)^{\frac{1}{2}}$.
20. $\sqrt[3]{8x - 24}$.
21. $\sqrt{16a^3 + 16a^2}$.
22. $(27c^6 - 27c^3)^{\frac{1}{3}}$.

Simplify:

23. $\sqrt{by^2 + 4by + 4b}$.

24. $\sqrt{5x^2 - 10xy + 5y^2}$.

25. $(3am^2 + 6am + 3a)^{\frac{1}{2}}$.

26. $(5a^4 + 10a^3x + 5a^2x^2)^{\frac{1}{2}}$.

27. Reduce $\sqrt{\frac{7}{12}}$ to its simplest form.

Solution. — To make the radicand integral and thus simplify it, we must multiply both its terms by a number that will make the denominator a perfect power corresponding to the index of the radical, in this case, a perfect square. The smallest multiplier that will accomplish this is 3, thus:

$$\sqrt{\frac{7}{12}} = \sqrt{\frac{7 \times 3}{12 \times 3}} = \sqrt{\frac{21}{36}} = \sqrt{\frac{1}{36} \times 21} = \sqrt{\frac{1}{36}} \times \sqrt{21} = \tfrac{1}{6}\sqrt{21}.$$

28. $\sqrt{\frac{1}{3}}$.

29. $\sqrt{\frac{4}{5}}$.

30. $\sqrt{\frac{5}{8}}$.

31. $\sqrt[3]{\frac{5}{6}}$.

32. $\sqrt[3]{\frac{3}{16}}$.

33. $\sqrt{\dfrac{3x^3}{y}}$.

34. $\sqrt[4]{\dfrac{a^5b}{c}}$.

35. $\sqrt[3]{\dfrac{2x^3y^2}{9b^2}}$.

36. $\sqrt{\dfrac{5}{8y^3}}$.

37. $\sqrt[3]{\dfrac{2a}{3b^2}}$.

38. $\sqrt[4]{\dfrac{xy^4}{125z^3}}$.

39. $(a + b)\sqrt{\dfrac{a + b}{a - b}}$.

40. $\dfrac{2y}{x - 2y}\sqrt{\dfrac{x - 2y}{2y}}$.

41. $\dfrac{(a + b)^2}{a - b}\sqrt[3]{\dfrac{a + b}{(a - b)^2}}$.

42. $(1 - x^3)\sqrt{\dfrac{1 - x + x^2}{1 + x + x^2}}$.

In general it may be proved that $a^{\frac{pm}{qm}} = a^{\frac{p}{q}}$; that is,

A number having a fractional exponent is not changed in value by reducing the fractional exponent to higher or lower terms.

43. Reduce $\sqrt[6]{4x^2y^4}$ to its simplest form.

Solution. $\sqrt[6]{4x^2y^4} = \sqrt[6]{(2xy^2)^2} = (2xy^2)^{\frac{2}{6}} = (2xy^2)^{\frac{1}{3}} = \sqrt[3]{2xy^2}$.

Simplify:

44. $\sqrt[4]{49}$.

45. $\sqrt[6]{27}$.

46. $\sqrt[4]{100}$.

47. $\sqrt[4]{400}$.

48. $\sqrt[6]{625}$.

49. $\sqrt[4]{576}$.

50. $\sqrt[6]{25a^4b^2}$.

51. $\sqrt[9]{27x^6y^3}$.

52. $\sqrt[6]{1000a^3}$.

53. $\sqrt[6]{a^4b^2x^4y^8}$.

54. $\sqrt[9]{125x^6z^3}$.

55. $\sqrt[4]{144a^2b^4c^2}$.

189. Reduction of a mixed surd to an entire surd.

EXERCISES

1. Express $2x\sqrt[3]{3y^2}$ as an entire surd.

SOLUTION. $2x\sqrt[3]{3y^2}=\sqrt[3]{8x^3}\sqrt[3]{3y^2}=\sqrt[3]{8x^3\times 3y^2}=\sqrt[3]{24x^3y^2}$.

RULE. — *Raise the coefficient to a power corresponding to the index of the given radical, and introduce the result under the radical sign as a factor.*

Express as entire surds:

2. $3\sqrt{2}$.
3. $2\sqrt[3]{2}$.
4. $5\sqrt{3}$.
5. $2\sqrt[4]{3}$.
6. $3\sqrt[3]{4}$.
7. $4\sqrt[4]{2}$.
8. $\frac{1}{3}\sqrt{6}$.
9. $\frac{1}{2}\sqrt[3]{12}$.
10. $a^2\sqrt[3]{b}$.
11. $\frac{1}{2}x\sqrt{x}$.
12. $\frac{3}{4}\sqrt{\frac{7}{9}}$.
13. $\frac{1}{3}b^2\sqrt[3]{36a}$.
14. $\frac{2}{3}\sqrt{1\frac{1}{2}}$.
15. $\frac{2}{3}\sqrt[4]{3\frac{3}{8}}$.
16. $\frac{3}{4}\sqrt[3]{7\frac{1}{9}}$.
17. $\frac{2}{a}\sqrt[3]{5a^2}$.
18. $\frac{1}{ab}(a-b)^{\frac{1}{2}}$.
19. $\frac{a+4}{a-4}\sqrt{1-\frac{8}{a+4}}$.

190. Reduction of radicals to the same order.

EXERCISES

1. Reduce $\sqrt[3]{2}$, $\sqrt{3}$, and $\sqrt[4]{5}$ to radicals of the same order.

SOLUTION.
$$\sqrt[3]{2}=2^{\frac{1}{3}}=2^{\frac{4}{12}}=\sqrt[12]{2^4}=\sqrt[12]{16}.$$
$$\sqrt{3}=3^{\frac{1}{2}}=3^{\frac{6}{12}}=\sqrt[12]{3^6}=\sqrt[12]{729}.$$
$$\sqrt[4]{5}=5^{\frac{1}{4}}=5^{\frac{3}{12}}=\sqrt[12]{5^3}=\sqrt[12]{125}.$$

RULE. — *Express the given radicals with fractional exponents having a common denominator. Raise each number to the power indicated by the numerator of its fractional exponent, and indicate the root expressed by the common denominator.*

Reduce to radicals of the same order:

2. $\sqrt{3}$ and $\sqrt[4]{2}$.
3. $\sqrt[3]{2}$ and $\sqrt{5}$.
4. $\sqrt{2}$ and $\sqrt[5]{4}$.
5. $\sqrt[3]{5}$ and $\sqrt[6]{4}$.
6. $\sqrt[3]{3}$ and $\sqrt[4]{2}$.
7. $\sqrt{6}$ and $\sqrt[3]{9}$.
8. $\sqrt[3]{4}$ and $\sqrt{10}$.
9. $\sqrt[4]{3}$ and $\sqrt[6]{11}$.
10. $\sqrt[3]{10}$ and $\sqrt[4]{12}$.

Reduce to radicals of the same order:

11. $\sqrt{3}$, $\sqrt[3]{2}$, and $\sqrt[6]{10}$.

12. $\sqrt[6]{12}$, $\sqrt[4]{3}$, and $\sqrt[3]{4}$.

13. $\sqrt[3]{5}$, $\sqrt{2}$, and $\sqrt[4]{10}$.

14. $\sqrt[5]{3}$, $\sqrt[4]{5}$, and $\sqrt[10]{12}$.

15. $\sqrt{\frac{2}{3}}$, $\sqrt[3]{\frac{1}{16}x}$, and $2\sqrt{5}$.

16. $\sqrt[n]{x}$, $\sqrt{xy}$, and $\sqrt[n]{x^2y^2}$.

17. $\sqrt[3]{a+b}$, $\sqrt{a-b}$, and $\sqrt[3]{a-b}$.

18. $\sqrt{a+b}$, $\sqrt[4]{a^2+b^2}$, and $\sqrt{a-b}$.

19. Which is greater, $\sqrt[4]{3}$ or $\sqrt{2}$? $\sqrt[6]{4}$ or $\sqrt[4]{2}$?

20. Which is greater, $\sqrt[5]{5}$ or $\sqrt[3]{3}$? $2\sqrt{3}$ or $3\sqrt[3]{2}$?

Arrange in order of decreasing value:

21. $\sqrt[6]{10}$, $\sqrt[3]{3}$, and $\sqrt{2}$.

22. $\sqrt{2}$, $\sqrt[5]{6}$, and $\sqrt[10]{33}$.

23. $\sqrt{3}$, $\sqrt[5]{7}$, and $\sqrt[4]{25}$.

24. $\sqrt[3]{3\frac{1}{2}}$, $\sqrt{3}$, $\sqrt[3]{5}$, and $\sqrt{2\frac{1}{2}}$.

25. $\sqrt[3]{4}$, $\sqrt{2}$, $\sqrt[4]{5}$, and $\sqrt[6]{13}$.

26. $2\sqrt{3}$, $3\sqrt[3]{2}$, $2\sqrt{5}$, and $\sqrt[3]{40}$.

Addition and Subtraction of Radicals

191. Principle. — *Only similar radicals can be united into one term by addition or subtraction.*

EXERCISES

192. Add:

1. $6\sqrt{\frac{1}{2}}$ and $4\sqrt[6]{8}$.

Solution

$$6\sqrt{\tfrac{1}{2}} = 3\sqrt{2}$$
$$4\sqrt[6]{8} = 4\sqrt{2}$$
$$\text{Sum} = 7\sqrt{2}$$

2. $\sqrt{32}$ and $\sqrt{72}$.

3. $\sqrt{48}$ and $\sqrt{108}$.

4. $\sqrt{8}$, $\sqrt{18}$, and $\sqrt[4]{324}$.

5. $\sqrt[3]{16}$, $2\sqrt[3]{\frac{1}{4}}$, and $\sqrt[3]{54}$.

6. $10\sqrt{\frac{1}{5}}$, $\sqrt{80}$, and $\sqrt[6]{125}$.

Subtract:

7. $\frac{1}{3}\sqrt[3]{24}$ from $\sqrt[3]{81}$.

Solution

$$\sqrt[3]{81} = 3\sqrt[3]{3}$$
$$\tfrac{1}{3}\sqrt[3]{24} = \tfrac{2}{3}\sqrt[3]{3}$$
$$\text{Difference} = 2\tfrac{1}{3}\sqrt[3]{3}$$

8. $\frac{1}{3}\sqrt{3}$ from $\sqrt{\frac{3}{4}}$.

9. $6\sqrt[3]{\frac{1}{4}}$ from $\sqrt[3]{128}$.

10. $\sqrt[3]{192}$ from $2\sqrt[3]{81}$.

11. $3\sqrt{\frac{3}{2}}$ from $2\sqrt[4]{36}$.

12. $\sqrt[3]{-32}$ from $\sqrt[3]{108}$.

Simplify:

13. $\sqrt[3]{135} - \sqrt[3]{625} + \sqrt[3]{320}$.

14. $\sqrt[3]{\frac{8}{5}} + \sqrt[3]{\frac{1}{5}} + \sqrt[3]{5\frac{2}{5}}$.

15. $\sqrt[3]{864} - \sqrt[3]{4000} + \sqrt[3]{32}$.

16. $2\sqrt{75} - 3\sqrt{72} + 5\sqrt{12}$.

17. $\sqrt{(x+y)^2a} - \sqrt{(x-y)^2a}$.

18. $\sqrt{\dfrac{x}{a^2}} - \sqrt{\dfrac{x}{b^2}} + \sqrt{\dfrac{x}{c^2}}$.

19. $\sqrt{\dfrac{x}{yz}} - \sqrt{\dfrac{y}{xz}} - \sqrt{\dfrac{z}{xy}}$.

20. $3\sqrt{\dfrac{3a}{4b^2}} + 2\sqrt{\dfrac{a}{3b^2}} - \sqrt{\dfrac{a}{27b^2}}$.

21. $\sqrt[3]{-24} + 3\sqrt[3]{-375} + \sqrt[3]{-81}$.

22. $\sqrt[5]{-96x^4} + 2\sqrt[5]{3x^4} - \sqrt[3]{5x} + \sqrt[3]{40x^4}$.

23. $\sqrt{3y^3 + 24y^2 + 48y} - \sqrt{3y^3 - 36y^2 + 108y}$.

24. $(\frac{8}{3})^{\frac{1}{2}} - (\frac{2}{3})^{-\frac{1}{2}} + \sqrt{(\frac{8}{27})^{-1}} + \sqrt{1.35} - \sqrt[4]{(1\frac{2}{3})^{-2}}$.

25. $5 \cdot 2^{-\frac{2}{3}} + 2^{-\frac{5}{3}} + 3 \cdot 2^{-\frac{8}{3}} + 3 \cdot 5^{-1} \cdot 2^{\frac{1}{5}} + \sqrt[5]{\frac{64}{3125}}$.

Multiplication of Radicals

193. Just as fractional exponents to be united by addition must be expressed with a common denominator, so radicals to be united by multiplication must be expressed with a common root index.

EXERCISES

194. 1. Multiply $3\sqrt{5}$ by $6\sqrt{8}$; $5\sqrt[3]{2}$ by $4\sqrt{3}$.

SOLUTIONS. $3\sqrt{5} \times 6\sqrt{8} = 18\sqrt{40} = 36\sqrt{10}$.

$5\sqrt[3]{2} \times 4\sqrt{3} = 5\sqrt[6]{4} \times 4\sqrt[6]{27} = 20\sqrt[6]{108}$.

RULE. — *If the radicals are not of the same order, reduce them to the same order.*

Multiply the coefficients for the coefficient of the product and the radicands for the radical factor of the product, and simplify the result, if necessary.

2. Multiply $\sqrt{12}$ by $\sqrt{3}$.

3. Multiply $\sqrt[3]{18}$ by $\sqrt[3]{3}$.

4. Multiply $2\sqrt{12}$ by $3\sqrt{6}$.

5. Multiply $3\sqrt{15}$ by $2\sqrt{5}$.

Find the value of:

6. $\sqrt{2} \times 3\sqrt[3]{3}$.
7. $2\sqrt[3]{9} \times 2\sqrt[3]{15}$.
8. $3\sqrt[4]{4} \times 2\sqrt{10}$.
9. $\sqrt[4]{5} \times \sqrt[6]{10}$.
10. $2\sqrt[3]{250} \times \sqrt{2}$.
11. $2\sqrt[5]{2} \times \sqrt[10]{512}$.
12. $\sqrt[3]{a^2b^2c^4} \times 3\sqrt{ab^4c^2}$.
13. $\sqrt{xy} \times 2\sqrt[3]{x^2y^2}$.
14. $\sqrt{\frac{2}{3}} \times \sqrt{\frac{4}{5}} \times \sqrt{\frac{5}{6}}$.
15. $\sqrt{\frac{1}{2}} \times \sqrt{\frac{7}{2}} \times \sqrt{\frac{4}{5}}$.
16. $\sqrt[3]{\frac{2}{3}} \times \sqrt{\frac{1}{2}} \times \sqrt[3]{\frac{3}{4}}$.
17. $\sqrt[3]{\frac{1}{3}} \times \sqrt[6]{\frac{3}{2}} \times \sqrt{\frac{2}{3}}$.
18. $2\sqrt{xy} \times \sqrt{3\,xy^2} \times \sqrt{6\,xy}$.
19. $2\sqrt{2\,ab} \times 3\sqrt[3]{a^2b} \times 3\sqrt[3]{8\,ab^2}$.
20. $\sqrt{x^{-1}y} \times \sqrt[3]{x^{-2}y^2} \times \sqrt{x^{-3}y^3}$.
21. $\sqrt[4]{(a+b)^2} \times \sqrt[8]{(a-b)^4} \times \sqrt[4]{(a+b)^{-2}}$.

Multiply:

22. $\sqrt{3}+\sqrt{2}$ by $\sqrt{3}-\sqrt{2}$.
23. $\sqrt{6}+\sqrt{5}$ by $\sqrt{5}-\sqrt{6}$.
24. $3\sqrt{7}+1$ by $3\sqrt{7}-1$.
25. $2\sqrt{3}+3\sqrt{5}$ by $3\sqrt{3}+2\sqrt{5}$.
26. $2\sqrt{6}-3\sqrt{5}$ by $4\sqrt{3}-\sqrt{10}$.
27. $x-\sqrt{xyz}+yz$ by $\sqrt{x}+\sqrt{yz}$.
28. $a+\sqrt{abc}+bc$ by $\sqrt{a}-\sqrt{bc}$.

Expand:

29. $(\sqrt{4+\sqrt{7}})(\sqrt{4-\sqrt{7}})$.
30. $(\sqrt{10+\sqrt{2}})(\sqrt{10-\sqrt{2}})$.
31. $(\sqrt{8+\sqrt{10}})(\sqrt{8-\sqrt{10}})$.
32. $(\sqrt{9\,y+y\sqrt{6}})(\sqrt{9\,y-y\sqrt{6}})$.
33. $(\sqrt{7\,c+\sqrt{5\,c^2}})(\sqrt{7\,c-\sqrt{5\,c^2}})$.
34. $(\sqrt{14\,x+x\sqrt{27}})(\sqrt{14\,x-x\sqrt{27}})$.

Division of Radicals

195. In division, when one fractional exponent is subtracted from another, the exponents must be expressed with a common denominator. When one radical is divided by another, the radicals must be expressed with a common root index.

EXERCISES

196. 1. Divide $2\sqrt[3]{4}$ by $4\sqrt{2}$; $\sqrt[3]{x^2}$ by $\sqrt[4]{y}$.

SOLUTION. $2\sqrt[3]{4} \div 4\sqrt{2} = 2\sqrt[6]{16} \div 4\sqrt[6]{8} = \frac{1}{2}\sqrt[6]{2}$.

$$\sqrt[3]{x^2} \div \sqrt[4]{y} = \frac{(x^2)^{\frac{1}{3}}}{y^{\frac{1}{4}}} = \frac{(x^8)^{\frac{1}{12}}}{(y^3)^{\frac{1}{12}}} = \left(\frac{x^8}{y^3}\right)^{\frac{1}{12}} = \sqrt[12]{\frac{x^8}{y^3}} = \frac{1}{y}\sqrt[12]{x^8y^9}.$$

RULE. — *If necessary, reduce the radicals to the same order.*

To the quotient of the coefficients annex the quotient of the radicands written under the common radical sign, and reduce the result to the simplest form, if necessary.

Find quotients:

2. $\sqrt[3]{54} \div \sqrt[3]{2}$.
3. $\sqrt{60} \div 2\sqrt{5}$.
4. $\sqrt{32} \div \sqrt{6}$.
5. $6\sqrt{5} \div \sqrt{20}$.
6. $12\sqrt{6} \div \sqrt{108}$.
7. $7\sqrt[3]{135} \div \sqrt[3]{9}$.
8. $6\sqrt{125} \div 5\sqrt{24}$.
9. $2\sqrt[4]{100} \div 4\sqrt{5}$.
10. $5\sqrt[3]{24} \div \sqrt[6]{625}$.
11. $2\sqrt{5} \div \sqrt[3]{5}$.
12. $3\sqrt[6]{216} \div \sqrt{12}$.
13. $2\sqrt[3]{12} \div \sqrt{8}$.
14. $\sqrt[3]{ax} \div \sqrt{xy}$.
15. $\sqrt{2xy^3} \div \sqrt[4]{x^2y^2}$.
16. $\sqrt{x+y} \div \sqrt{x-y}$.
17. $\sqrt[3]{4x^2y^2} \div \sqrt{2xy}$.
18. $\sqrt[4]{9a^2b^2} \div \sqrt[3]{3ab}$.
19. $\sqrt[3]{2xy} \div \sqrt[8]{4x^2y^2}$.
20. $\sqrt{\frac{1}{4}} \div \sqrt[6]{\frac{1}{8}}$.
21. $\sqrt[3]{\frac{3}{4}} \div \frac{1}{2}\sqrt[6]{\frac{4}{5}}$.
22. $3\sqrt[3]{\frac{2}{3}} \div \sqrt{\frac{3}{4}}$.
23. $2\sqrt[4]{\frac{9}{16}} \div 6\sqrt[3]{3}$.
24. $(\sqrt{30} - \sqrt{5}) \div \sqrt{5}$.
25. $(6 - 3\sqrt{2} + \sqrt{12}) \div \sqrt{3}$.
26. $(4\sqrt{3} + 4\sqrt{2}) \div (\sqrt{6} + 2)$.
27. $(12 + 8\sqrt{6}) \div (\sqrt{3} + 2\sqrt{2})$.

Involution and Evolution of Radicals

197. In finding powers and roots of radicals, it is frequently convenient to use fractional exponents.

EXERCISES

198. 1. Find the cube of $3\sqrt{x}$.

SOLUTION. $(3\sqrt{x})^3 = (3x^{\frac{1}{2}})^3 = 3^3x^{\frac{3}{2}} = 27x^{\frac{3}{2}} = 27\sqrt{x^3} = 27x\sqrt{x}.$

2. Find the square of $3\sqrt[4]{16}$.

SOLUTION. $(3\sqrt[4]{16})^2 = (3 \cdot 16^{\frac{1}{4}})^2 = 3^2 \cdot 16^{\frac{1}{2}} = 9 \cdot 4 = 36.$

Square:	Cube:	Involve as indicated:
3. $2\sqrt{xy}$.	7. $2\sqrt{2}$.	11. $(3\sqrt{xy})^4$.
4. $3\sqrt[3]{2a}$.	8. $3\sqrt[3]{x^2}$.	12. $(-2\sqrt[6]{2a})^3$.
5. $b\sqrt[3]{3b^2}$.	9. $\sqrt[4]{c^3d^3}$.	13. $(-3a^{\frac{m}{2}}x^{\frac{n}{3}})^6$.
6. $y^2\sqrt[4]{4c^2d}$.	10. $\sqrt[6]{4n^3}$.	14. $(-2\sqrt{a}\sqrt[3]{b})^5$.

15. Find the square root of $\sqrt[3]{27x}$.

SOLUTION. $\sqrt{\sqrt[3]{27x}} = [(27x)^{\frac{1}{3}}]^{\frac{1}{2}} = (27x)^{\frac{1}{6}} = \sqrt[6]{27x}.$

Find the square root of:		Find the cube root of:	
16. $\sqrt{ab}$.	19. $\sqrt[3]{x^n}$.	22. $\sqrt{3a}$.	25. $-8\sqrt{a^6}$.
17. $\sqrt[3]{3c}$.	20. $\sqrt[5]{a^4b^4}$.	23 $\sqrt{7a^3}$.	26. $-\sqrt{a^nb^n}$.
18. $\sqrt[4]{4a^2}$.	21. $\sqrt[n]{c^nx^2}$.	24. $\sqrt[3]{ab^3x}$.	27. $-27\sqrt[5]{x^3z^3}$.

Simplify these indicated roots:

28. $\sqrt[3]{\sqrt{a^6x^4}}$. 29. $(\sqrt[3]{4x^4y^4})^{\frac{1}{2}}$. 30. $\sqrt[4]{\sqrt[n]{a^4x^8}}$. 31. $(\sqrt{x^my^m})^{\frac{1}{mn}}$.

199. **Square root of binomial quadratic surds by inspection.**

1. Define **binomial surd; binomial quadratic surd; conjugate surds.**

2. Since $(\sqrt{2}+\sqrt{3})^2 = 2 + 2\sqrt{6} + 3 = 5 + 2\sqrt{6}$, the terms of the square root of $5 + 2\sqrt{6}$ may be obtained by separating $\sqrt{6}$ into two factors such that the sum of their squares is 5; they are $\sqrt{2}$ and $\sqrt{3}$; then, the square root of $5 + 2\sqrt{6}$ is $\sqrt{2}+\sqrt{3}$. That is,

200. Principle. — *The terms of the square root of a binomial quadratic surd that is a perfect square may be obtained by dividing the irrational term by 2 and then separating the quotient into two factors, the sum of whose squares is the rational term.*

EXERCISES

201. **1.** Find the square root of $12 - 6\sqrt{3}$.

Solution. $12 - 6\sqrt{3} = 12 - 2(3\sqrt{3}) = 12 - 2\sqrt{27}$.

Since $\sqrt{27} = \sqrt{9} \times \sqrt{3}$ and $12 = 9 + 3$,

$\sqrt{12 - 6\sqrt{3}} = \sqrt{9} - \sqrt{3} = 3 - \sqrt{3}$.

Find the square root of:

2. $5 - 2\sqrt{6}$.

3. $8 + 2\sqrt{15}$.

4. $9 - 2\sqrt{14}$.

5. $11 - 2\sqrt{30}$.

6. $11 - 6\sqrt{2}$.

7. $22 + 8\sqrt{6}$.

8. $24 - 8\sqrt{5}$.

9. $31 + 12\sqrt{3}$.

10. $3 - 2\sqrt{2}$.

11. $6 + 2\sqrt{5}$.

12. $a^2 + b + 2a\sqrt{b}$.

13. $2a - 2\sqrt{a^2 - b^2}$.

202. **Square root of binomial quadratic surds by conjugate relations.**

Principles. — 1. *The square root of a rational number cannot be partly rational and partly a quadratic surd.*

For, if possible, let $\sqrt{y} = \sqrt{b} \pm m$, $\sqrt{y}$ and $\sqrt{b}$ being surds.

By squaring, $y = b \pm 2m\sqrt{b} + m^2$,

and $$\sqrt{b} = \pm \frac{y - m^2 - b}{2m},$$

which is impossible, because a surd cannot be equal to a rational number.

Therefore, $\sqrt{y}$ cannot be equal to $\sqrt{b} \pm m$.

2. *In any equation containing rational numbers and quadratic surds, as* $a + \sqrt{b} = x + \sqrt{y}$, *the rational parts are equal, and also the irrational parts.*

Given $$a + \sqrt{b} = x + \sqrt{y}. \qquad (1)$$

Since a and x are both rational, if possible, let

$$a = x \pm m. \qquad (2)$$

Then, $$x \pm m + \sqrt{b} = x + \sqrt{y}, \qquad (3)$$

and $$\sqrt{y} = \sqrt{b} \pm m. \qquad (4)$$

Since, Prin. 1, equation (4) is impossible, $a = x \pm m$ is impossible; that is, a is neither greater nor less than x.

Therefore, $a = x$, and from (1), $\sqrt{b} = \sqrt{y}$.

Hence, if $a + \sqrt{b} = x + \sqrt{y}$, $a = x$ and $\sqrt{b} = \sqrt{y}$.

3. *If* $a + \sqrt{b}$ *and* $a - \sqrt{b}$ *are binomial quadratic surds and* $\sqrt{a + \sqrt{b}} = \sqrt{x} + \sqrt{y}$, *then* $\sqrt{a - \sqrt{b}} = \sqrt{x} - \sqrt{y}$.

To exclude imaginary numbers, suppose that $a - \sqrt{b}$ is positive.

Given $$\sqrt{a + \sqrt{b}} = \sqrt{x} + \sqrt{y}.$$

Square, Ax. 6, $$a + \sqrt{b} = x + 2\sqrt{xy} + y.$$

Therefore, Prin. 2, $$a = x + y \text{ and } \sqrt{b} = 2\sqrt{xy};$$

whence, Ax. 2, $$a - \sqrt{b} = x + y - 2\sqrt{xy}.$$

Hence, Ax. 7, $$\sqrt{a - \sqrt{b}} = \sqrt{x} - \sqrt{y}.$$

EXERCISES

203. 1. Find the square root of $21 + 6\sqrt{10}$.

Solution. — Let $$\sqrt{x} + \sqrt{y} = \sqrt{21 + 6\sqrt{10}}. \qquad (1)$$

Then, Prin. 3, $$\sqrt{x} - \sqrt{y} = \sqrt{21 - 6\sqrt{10}}. \qquad (2)$$

Multiply (1) by (2), $$x - y = \sqrt{441 - 360} = \sqrt{81},$$

or $$x - y = 9. \qquad (3)$$

Square (1), Ax. 6, $$x + 2\sqrt{xy} + y = 21 + 6\sqrt{10}.$$

Therefore, Prin. 2, $$x + y = 21. \qquad (4)$$

Solve (4) and (3), $$x = 15,\ y = 6.$$

$$\therefore \sqrt{x} = \sqrt{15},\ \sqrt{y} = \sqrt{6}.$$

Hence, from (1), $$\sqrt{21 + 6\sqrt{10}} = \sqrt{15} + \sqrt{6}.$$

Find the square root of:

2. $25+10\sqrt{6}$.
3. $19+6\sqrt{2}$.
4. $45+30\sqrt{2}$.
5. $35-14\sqrt{6}$.
6. $11+6\sqrt{2}$.
7. $24-8\sqrt{5}$.
8. $16+6\sqrt{7}$.
9. $21-8\sqrt{5}$.
10. $47-12\sqrt{11}$.
11. $56+32\sqrt{3}$.
12. $35-12\sqrt{6}$.
13. $56-12\sqrt{3}$.
14. $2+\sqrt{3}$.
15. $6+\sqrt{35}$.
16. $1+\frac{2}{3}\sqrt{2}$.
17. $2+\frac{4}{5}\sqrt{6}$.
18. $18-6\sqrt{5}$.
19. $30+20\sqrt{2}$.

Rationalization

204. If it is required to find the approximate value of $\frac{1}{\sqrt{5}}$, we may divide 1 by the approximate square root of 5, using long division, but it will be more accurate and a saving of labor to change the fraction to an equivalent fraction having a *rational* denominator, thus,

$$\frac{1}{\sqrt{5}}=\frac{1\cdot\sqrt{5}}{\sqrt{5}\cdot\sqrt{5}}=\frac{\sqrt{5}}{5},$$

and divide $\sqrt{5}$ by the simple and *rational* divisor 5.

205. Define **rationalization**; **rationalizing factor**; **rationalizing the denominator.**

EXERCISES

206. Rationalize the denominator of:

1. $\frac{1}{\sqrt{a^3}}$.
2. $\frac{bc}{\sqrt{abc^3}}$.
3. $\frac{2x^2}{\sqrt[3]{x^2y^2}}$.
4. $\frac{\sqrt[3]{6}}{\sqrt{12}}$.
5. $\sqrt[3]{\frac{a^2b}{2ax^2}}$.
6. $\frac{\sqrt{x-y}}{\sqrt{x+y}}$.
7. $\sqrt[3]{\frac{a+2}{(a-2)^2}}$.
8. $\sqrt{1-\frac{4}{x+2}}$.

Taking $\sqrt{2}=1.414$, $\sqrt{3}=1.732$, and $\sqrt{5}=2.236$, find, to the nearest hundredth, the value of:

9. $\frac{3}{\sqrt{2}}$.
10. $\frac{5}{\sqrt{3}}$.
11. $\frac{7}{\sqrt{5}}$.
12. $\frac{12}{\sqrt{8}}$.
13. $\frac{20}{\sqrt{32}}$.
14. $\frac{18}{\sqrt{27}}$.
15. $\frac{10}{\sqrt[6]{125}}$.
16. $\frac{21}{\sqrt[4]{324}}$.

207. Since, § 30, 3, $(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})=(\sqrt{a})^2-(\sqrt{b})^2$ $=a-b$, the product of any two conjugate surds is rational. Hence,

Principle. — *A binomial quadratic surd may be rationalized by multiplying it by its conjugate.*

EXERCISES

208. 1. Rationalize the denominator of $\dfrac{\sqrt{7}-\sqrt{3}}{\sqrt{7}+\sqrt{3}}$.

Solution

$$\frac{\sqrt{7}-\sqrt{3}}{\sqrt{7}+\sqrt{3}}=\frac{(\sqrt{7}-\sqrt{3})(\sqrt{7}-\sqrt{3})}{(\sqrt{7}+\sqrt{3})(\sqrt{7}-\sqrt{3})}=\frac{7-2\sqrt{21}+3}{7-3}=\frac{5-\sqrt{21}}{2}.$$

Rationalize the denominator of:

2. $\dfrac{4}{3+\sqrt{2}}$.

3. $\dfrac{c}{\sqrt{a}-\sqrt{b}}$.

4. $\dfrac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}$.

5. $\dfrac{2-\sqrt{2}}{5-3\sqrt{2}}$.

6. $\dfrac{\sqrt{a}+\sqrt{x}}{\sqrt{a}-\sqrt{x}}$.

7. $\dfrac{a+2\sqrt{b}}{a-2\sqrt{b}}$.

8. $\dfrac{2-\sqrt{3}}{2\sqrt{3}+1}$.

9. $\dfrac{3\sqrt{3}-2\sqrt{2}}{4\sqrt{2}+6\sqrt{3}}$.

10. $\dfrac{x-\sqrt{x^2-1}}{x+\sqrt{x^2-1}}$.

11. $\dfrac{\sqrt{a^2+a+1}-1}{\sqrt{a^2+a+1}+1}$.

12. $\dfrac{\sqrt{a+b}-\sqrt{a-b}}{\sqrt{a+b}+\sqrt{a-b}}$.

13. $\dfrac{\sqrt{x^2-2}-\sqrt{x^2+2}}{\sqrt{x^2-2}+\sqrt{x^2+2}}$.

Reduce to a decimal, to the nearest thousandth:

14. $\dfrac{4}{2+\sqrt{3}}$.

15. $\dfrac{8}{3-\sqrt{5}}$.

16. $\dfrac{3-\sqrt{2}}{2-\sqrt{2}}$.

17. $\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$.

18. $\dfrac{5-\sqrt{6}}{3-\sqrt{6}}$.

19. $\dfrac{2-\sqrt{3}}{4+2\sqrt{3}}$.

20. Rationalize the denominator of $\dfrac{\sqrt{2}-\sqrt{3}-\sqrt{5}}{\sqrt{2}+\sqrt{3}+\sqrt{5}}$.

SOLUTION. $$\frac{\sqrt{2}-\sqrt{3}-\sqrt{5}}{\sqrt{2}+\sqrt{3}+\sqrt{5}}=\frac{(\sqrt{2}-\sqrt{5})-\sqrt{3}}{(\sqrt{2}+\sqrt{3})+\sqrt{5}}\times\frac{(\sqrt{2}-\sqrt{5})+\sqrt{3}}{(\sqrt{2}+\sqrt{3})-\sqrt{5}}$$
$$=\frac{2-2\sqrt{10}+5-3}{2+2\sqrt{6}+3-5}=\frac{4-2\sqrt{10}}{2\sqrt{6}}$$
$$=\frac{2-\sqrt{10}}{\sqrt{6}}=\frac{2\sqrt{6}-2\sqrt{15}}{6}=\frac{\sqrt{6}-\sqrt{15}}{3}.$$

Rationalize the denominator of:

21. $\dfrac{\sqrt{2}-\sqrt{5}-\sqrt{7}}{\sqrt{2}+\sqrt{5}+\sqrt{7}}$.

22. $\dfrac{1}{\sqrt{2}+\sqrt{3}+\sqrt{5}}$.

23. $\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{3}+\sqrt{2}-\sqrt{6}}$.

24. $\dfrac{2\sqrt{2}-3\sqrt{3}+4\sqrt{5}}{\sqrt{2}+\sqrt{3}-\sqrt{5}}$.

25. Rationalize the denominator of $\dfrac{a}{\sqrt{a}+\sqrt[3]{b^2}}$, or $\dfrac{a}{a^{\frac{1}{2}}+b^{\frac{2}{3}}}$.

SOLUTION. — By § 38, $\sqrt{a}+\sqrt[3]{b^2}$, or $a^{\frac{1}{2}}+b^{\frac{2}{3}}$, is exactly contained in the sum of any like odd powers of $a^{\frac{1}{2}}$ and $b^{\frac{2}{3}}$, and also in the difference of any like even powers of $a^{\frac{1}{2}}$ and $b^{\frac{2}{3}}$. The lowest like powers of $a^{\frac{1}{2}}$ and $b^{\frac{2}{3}}$ that are rational numbers are the sixth powers, which are even powers.

Hence, the rational expression of lowest degree in which $a^{\frac{1}{2}}+b^{\frac{2}{3}}$ is exactly contained is $(a^{\frac{1}{2}})^6-(b^{\frac{2}{3}})^6$, or a^3-b^4.

Dividing a^3-b^4 by $a^{\frac{1}{2}}+b^{\frac{2}{3}}$, we find that the rationalizing factor for the denominator is $a^{\frac{5}{2}}-a^2b^{\frac{2}{3}}+a^{\frac{3}{2}}b^{\frac{4}{3}}-ab^2+a^{\frac{1}{2}}b^{\frac{8}{3}}-b^{\frac{10}{3}}$.

Multiplying both terms of the given fraction by this factor, we have

$$\frac{a}{\sqrt{a}+\sqrt[3]{b^2}},\text{ or }\frac{a}{a^{\frac{1}{2}}+b^{\frac{2}{3}}}=\frac{a(a^{\frac{5}{2}}-a^2b^{\frac{2}{3}}+a^{\frac{3}{2}}b^{\frac{4}{3}}-ab^2+a^{\frac{1}{2}}b^{\frac{8}{3}}-b^{\frac{10}{3}})}{a^3-b^4}.$$

Rationalize the denominator of:

26. $\dfrac{\sqrt[3]{ab}}{\sqrt[3]{a}-\sqrt[3]{b}}$.

27. $\dfrac{2}{\sqrt[3]{x}+\sqrt{y}}$.

28. $\dfrac{\sqrt[3]{ab^2}}{\sqrt[3]{a^2}-\sqrt{b^3}}$.

29. $\dfrac{\sqrt{a}+b}{\sqrt[4]{a}-\sqrt{b}}$.

30. $\dfrac{\sqrt{ax}}{\sqrt[3]{a}-\sqrt[5]{x}}$.

31. $\dfrac{\sqrt[3]{xy^2}}{\sqrt{x}+\sqrt[5]{y^3}}$.

Radical Equations

209. When the following equations have been freed of radicals, the resulting equations will be found to be simple equations. Other varieties of **radical**, or **irrational, equations** are treated later.

EXERCISES

210. 1. Solve the equation $\sqrt{x-5}+\sqrt{x}=5$.

SOLUTION. $\sqrt{x-5}+\sqrt{x}=5.$

Transpose $\sqrt{x}$, $\sqrt{x-5}=5-\sqrt{x}.$

Square, Ax. 6, $x-5=25-10\sqrt{x}+x.$

Transpose and combine, $10\sqrt{x}=30.$

Divide by 10, $\sqrt{x}=3.$

Square, $x=9.$

VERIFICATION. $\sqrt{9-5}+\sqrt{9}=\sqrt{4}+\sqrt{9}=2+3=5$; that is, $5=5$.

2. Given $\sqrt{14+\sqrt{1+\sqrt{x+8}}}=4$, to find the value of x.

SOLUTION. $\sqrt{14+\sqrt{1+\sqrt{x+8}}}=4.$

Square, $14+\sqrt{1+\sqrt{x+8}}=16.$

Transpose, etc., $\sqrt{1+\sqrt{x+8}}=2.$

Square, $1+\sqrt{x+8}=4.$

Transpose, etc., $\sqrt{x+8}=3.$

Square, $x+8=9.$

$\therefore x=1.$

VERIFICATION. $\sqrt{14+\sqrt{1+\sqrt{1+8}}}=\sqrt{14+\sqrt{1+3}}$

$=\sqrt{14+2}=4$; that is, $4=4$.

General directions. — *Transpose so that the radical term, if there is but one, or the most complex radical term, if there is more than one, may constitute one member of the equation.*

Then raise each member to a power corresponding to the order of that radical and simplify.

If the equation is not freed of radicals by the first involution, proceed again as at first.

Solve for x, and verify each result:

3. $\sqrt{x} - 2 = 10.$
4. $3 + 2\sqrt{x} = 15.$
5. $3\sqrt{2x - 4} = 32.$
6. $\sqrt{x + 11} = 4.$
7. $\sqrt{x + 5} = 13.$
8. $\sqrt{x + d^2} = c.$
9. $\sqrt[3]{x - 2} = 2.$
10. $\sqrt[3]{x + b^3} = a.$
11. $\sqrt[3]{x} + b = a.$
12. $\sqrt[3]{4x - 16} = 2.$
13. $1 + 2\sqrt{x} = 7 - \sqrt{x}.$
14. $\sqrt{x - 21} = \sqrt{x} - 1.$
15. $\sqrt{x^2 - 11} + 1 = x.$
16. $\sqrt{x - 16} = 8 - \sqrt{x}.$
17. $\sqrt{x - 15} + \sqrt{x} = 15.$
18. $\sqrt{2x} - \sqrt{2x - 15} = 1.$
19. $\sqrt{x} + 2 = \sqrt{x + 32}.$
20. $\sqrt{x + 4} = 4 - \sqrt{x - 4}.$
21. $\sqrt{x - 5} + \sqrt{x + 7} = 6.$
22. $\sqrt{x} - \sqrt{4x - 21} = 0.$
23. $\sqrt{9x + 8} + \sqrt{9x} - 4 = 0.$
24. $2\sqrt{x^2 + x + 1} = 2(2 + x).$
25. $3 - \sqrt{3 - 6x + 4x^2} = 2x.$
26. $\sqrt{2(1 - x)(3 - 2x)} - 1 = 2x.$
27. $\sqrt{16x + 3} + \sqrt{16x + 8} = 5.$
28. $\sqrt{1 + x\sqrt{x^2 + 12}} = 1 + x.$
29. $x + \sqrt{x^2 + \sqrt{2 + 4x^2}} = 1.$
30. $\sqrt{3(x + 1)} + \sqrt{3x - 1} = \sqrt{2(6x + 1)}.$
31. $2\sqrt{x} - \sqrt{4x - 22} - \sqrt{2} = 0.$
32. $\sqrt{9x^2 - 4\sqrt{9x^2 - 2}} + 3x = 2.$
33. $\sqrt{\sqrt{\sqrt{2x + 56}}} = 2.$
34. $\sqrt{7 + \sqrt{1 + \sqrt{4 + \sqrt{1 + 2\sqrt{x}}}}} = 3.$
35. $\sqrt{3x + 7} + \sqrt{4x - 3} = \sqrt{4x + 4} + \sqrt{3x}.$

Solve and verify:

36. $\dfrac{5}{\sqrt{3x+2}}=\sqrt{3x+2}+\sqrt{3x-1}.$

SUGGESTION. — Clear the equation of fractions.

37. $\dfrac{\sqrt{x}+12}{\sqrt{x}+5}=\dfrac{\sqrt{x}+8}{\sqrt{x}+3}.$

SUGGESTION. — Reduce each fraction to a mixed number and simplify before clearing of fractions.

38. $\dfrac{\sqrt{v}-6}{\sqrt{v}-1}=\dfrac{\sqrt{v}-8}{\sqrt{v}-5}.$

39. $\dfrac{\sqrt{2r}+6}{\sqrt{2r}+4}=\dfrac{\sqrt{2r}+2}{\sqrt{2r}+1}.$

40. $\dfrac{\sqrt{11n}+\sqrt{2n+3}}{\sqrt{11n}-\sqrt{2n+3}}=\dfrac{8}{3}.$

41. $\dfrac{2\sqrt{2x}+4}{2\sqrt{2x}-4}=\dfrac{\sqrt{x+1}+3}{\sqrt{x+1}-3}.$

42. $\dfrac{\sqrt{m+1}-\sqrt{m-1}}{\sqrt{m+1}+\sqrt{m-1}}=\dfrac{1}{2}.$

43. $\dfrac{\sqrt{4z+3}+2\sqrt{z-1}}{\sqrt{4z+3}-2\sqrt{z-1}}=5.$

44. $\dfrac{\sqrt{\sqrt{5x}-9}}{\sqrt{\sqrt{5x}+11}}=\dfrac{\sqrt{\sqrt{5x}-21}}{\sqrt{\sqrt{5x}-16}}.$

SUGGESTION. — First square both members.

45. $\dfrac{\sqrt{\sqrt{8x}+16}}{\sqrt{\sqrt{8x}+4}}=\dfrac{\sqrt{\sqrt{8x}+32}}{\sqrt{\sqrt{8x}+12}}.$

46. $\dfrac{x-3}{\sqrt{x}-\sqrt{3}}=\dfrac{\sqrt{x}+\sqrt{3}}{2}+2\sqrt{3}.$

SUGGESTION. — Simplify the first member.

47. $\sqrt{2x}-\sqrt{2x-7}=\dfrac{3}{\sqrt{2x-7}}.$

48. Solve $\dfrac{\sqrt{x+a}+\sqrt{x-a}}{\sqrt{x+a}-\sqrt{x-a}}=2+\dfrac{\sqrt{x^2-a^2}}{a}$ for x.

SUGGESTION. — Rationalize the denominator of the first fraction.

49. Solve $\dfrac{a+x+\sqrt{2ax+x^2}}{a+x-\sqrt{2ax+x^2}}=b^2$ for x.

Solve for x, and verify:

50. $\sqrt{x}+\sqrt{x-(a-b)^2}=a+b.$

51. $a\sqrt{x}-b\sqrt{x}=a^2+b^2-2\,ab.$

52. $\sqrt{5\,ax-9\,a^2}+a=\sqrt{5\,ax}.$

53. $\sqrt{x+3\,a}=\dfrac{6\,a}{\sqrt{x+3\,a}}-\sqrt{x}.$

54. $\sqrt{x}+\sqrt{2\,x}+\sqrt{3\,x}=\sqrt{a}.$

SOLUTION. — Factor, $(\sqrt{1}+\sqrt{2}+\sqrt{3})\sqrt{x}=\sqrt{a}.$

Multiply by $1+\sqrt{2}-\sqrt{3}$ to partially rationalize the first factor,

$$(1+2\sqrt{2}+2-3)\sqrt{x}=\sqrt{a}(1+\sqrt{2}-\sqrt{3}),$$

or
$$2\sqrt{2}\cdot\sqrt{x}=\sqrt{a}(1+\sqrt{2}-\sqrt{3}).$$

Square,
$$8\,x=a(1+\sqrt{2}-\sqrt{3})^2;$$

whence,
$$x=\frac{a}{8}(1+\sqrt{2}-\sqrt{3})^2.$$

Solve for x, giving the result with a rational denominator:

55. $\sqrt{2\,x}+\sqrt{3\,x}+\sqrt{5\,x}=\sqrt{m}.$

56. $\sqrt{2\,x}+\sqrt{3\,x}-\sqrt{5\,x}=\sqrt{c}.$

57. $\sqrt{x-a}+\sqrt{2(x-a)}=\sqrt{3\,x+a\sqrt{2}}.$

211. The student will have observed that radical equations are freed of radicals either by *rationalization* or by *involution*.

Thus,	$\sqrt{2\,x}-6=0$ (1)	$\sqrt{2\,x}+6=0$ (2)
Multiply by	$\sqrt{2\,x}+6$	$\sqrt{2\,x}-6$
	$2\,x-36=0$	$2\,x-36=0$
	$\therefore x=18$	$\therefore x=18$

If the positive, or principal, square root of $2\,x$ is taken, $x=18$ satisfies (1) but not (2); if the negative square root of $2\,x$ is taken, $x=18$ satisfies (2) but not (1).

It has been agreed, however, that the sign $\sqrt{\ }$ shall denote only principal roots in this chapter, and because of this arbitrary *convention*, our conclusion must be that (1) has the root $x=18$ and that (2) has no root, or is *impossible*.

According to this view, when both members of (1) are multiplied by $\sqrt{2x}+6$, no root is introduced because $\sqrt{2x}+6=0$ has no root; but when both members of (2), which has no root, are multiplied by $\sqrt{2x}-6$, the root of $\sqrt{2x}-6=0$, which is $x=18$, is introduced (§ 108).

A root may be introduced in this way by *rationalization*, or by the equivalent process of *squaring*.

Thus, $\sqrt{2x}+6=0.$ (2)

Transposing, we have $\sqrt{2x}=-6.$

Squaring, Ax. 6, we have $2x=36.$

$\therefore x=18.$

Verifying, we have $\sqrt{2\cdot 18}+6=6+6\neq 0.$

EXERCISES

212. 1. Solve, if possible, the equation

$$\sqrt{x-7}-\sqrt{x}=7.$$

Solution. — Transposing, squaring, simplifying, etc., we have

$$\sqrt{x}=-4.$$

Squaring, we have $x=16.$

Verification. $\sqrt{16-7}-\sqrt{16}=\sqrt{9}-\sqrt{16}=3-4\neq 7.$

Hence, the equation has no root, or is *impossible*.

Solve, and verify to discover which of the following equations are impossible; then change these to true equations:

2. $\sqrt{2x}+\sqrt{2x-3}=1.$

3. $\sqrt{3x+7}+\sqrt{3x}=7.$

4. $2\sqrt{x}+\sqrt{4x-11}=1.$

5. $\sqrt{4x+5}-2\sqrt{x-1}=9.$

6. $\sqrt{4x}-\sqrt{x}=\sqrt{9x-32}.$

7. $\sqrt{5x-1}-1=\sqrt{5x+16}.$

8. $\sqrt{x+1}+\sqrt{x+2}-\sqrt{4x+5}=0.$

9. $\sqrt{2(x^2+3x-5)}=(x+2)\sqrt{2}.$

10. $\dfrac{\sqrt{x-5}}{\sqrt{x-4}}+\dfrac{\sqrt{x+1}}{\sqrt{x+8}}=0.$

11. $\dfrac{\sqrt{19x}+\sqrt{2x+11}}{\sqrt{19x}-\sqrt{2x+11}}=2\frac{1}{6}.$

Jan. 16, 1919.

IMAGINARY NUMBERS

213. Our number system now comprises **natural numbers**, 1, 2, 3, ... ; **fractions**, arising from the indicated division of one natural number by another; **negative numbers** (denoting opposition to positive numbers), arising from the subtraction of a number from a less number; **surds**, arising from the attempt to extract a root of a number that is not a perfect power; and finally **imaginary numbers**, arising from the attempt to extract an even root of a negative number.

In this chapter only imaginary numbers of the second order will be treated.

Before the introduction of imaginary numbers, the only numbers known were those *whose squares are positive*, now called **real numbers** to distinguish them from **imaginary numbers**, *whose squares are negative.*

214. Since the square of an imaginary number is negative, imaginary numbers present an apparent exception, *in regard to signs*, to the distributive law for evolution. Apparently

$\sqrt{-1} \times \sqrt{-1}$ would equal $\sqrt{(-1)(-1)} = \sqrt{+1} = \pm 1.$

But by the definition of a root, the square of the square root of a number is the number itself.

$$\text{Hence,} \quad \sqrt{-1} \times \sqrt{-1} = (\sqrt{-1})^2 = -1, \text{ not } +1. \qquad (A)$$

In this chapter it will be assumed that imaginary numbers obey the same laws as real numbers, the signs being determined by (A), which we call the **fundamental property of imaginaries.**

215. Powers of $\sqrt{-1}$.

$$(\sqrt{-1}) = +\sqrt{-1};$$

$$(\sqrt{-1})^2 = (\sqrt{-1})(\sqrt{-1}) = -1;$$

$$(\sqrt{-1})^3 = (\sqrt{-1})^2\sqrt{-1} = (-1)\sqrt{-1} = -\sqrt{-1};$$

$$(\sqrt{-1})^4 = (\sqrt{-1})^2(\sqrt{-1})^2 = (-1)(-1) = +1;$$

$$(\sqrt{-1})^5 = (\sqrt{-1})^4\sqrt{-1} = (+1)\sqrt{-1} = +\sqrt{-1};$$

and so on. Hence, if $n = 0$ or a positive integer,

$$\left.\begin{aligned} (\sqrt{-1})^{4n+1} = +\sqrt{-1};\ (\sqrt{-1})^{4n+2} = -1; \\ (\sqrt{-1})^{4n+3} = -\sqrt{-1};\ (\sqrt{-1})^{4n+4} = +1. \end{aligned}\right\} \qquad (B)$$

Hence, *any even power of* $\sqrt{-1}$ *is real and any odd power is imaginary.*

For brevity $\sqrt{-1}$ is often written i.

216. Operations involving imaginary numbers.

EXERCISES

Find the value of:

1. $(\sqrt{-1})^6$.
3. $(\sqrt{-1})^{10}$.
5. $(\sqrt{-1})^{18}$.
7. $(-i)^3$.
2. $(\sqrt{-1})^7$.
4. $(\sqrt{-1})^{21}$.
6. $(\sqrt{-1})^{15}$.
8. $(-i)^8$.
9. Add $\sqrt{-a^4}$ and $\sqrt{-16\,a^4}$.

Solution

$$\sqrt{-a^4} + \sqrt{-16\,a^4} = a^2\sqrt{-1} + 4\,a^2\sqrt{-1} = 5\,a^2\sqrt{-1}.$$

Simplify:

10. $\sqrt{-4} + \sqrt{-49}$.
13. $\sqrt{-12} + 4\sqrt{-3}$.
11. $\sqrt{-9} + \sqrt{-64}$.
14. $5\sqrt{-18} - \sqrt{-72}$.
12. $2\sqrt{-4} + 3\sqrt{-1}$.
15. $3\sqrt{-20} - \sqrt{-80}$.

Simplify:

16. $\sqrt{-16a^2x^2}+\sqrt{-a^2x^2}-\sqrt{-9a^2x^2}$.

17. $(\sqrt{-a}+3\sqrt{-b})+(\sqrt{-a}-3\sqrt{-b})$.

18. $(\sqrt{-9xy}-\sqrt{-xy})-(\sqrt{-4xy}+\sqrt{-xy})$.

19. $\sqrt{-x^2}+\sqrt{-4x^2}-\sqrt{-x^3}+3x\sqrt{-x}$.

20. $\sqrt{-16}-3\sqrt{-4}+\sqrt{-18}+\sqrt{-50}+\sqrt{-25}$.

21. $\sqrt{-8}+a\sqrt{-2}-\sqrt{-98}-5\sqrt{-2a^2}$.

22. $\sqrt{1-5}-3\sqrt{1-10}+2\sqrt{5-30}$.

23. Multiply $3\sqrt{-10}$ by $2\sqrt{-8}$.

PROCESS

$$\begin{aligned}3\sqrt{-10}\times 2\sqrt{-8} &= 3\sqrt{10}\sqrt{-1}\times 2\sqrt{8}\sqrt{-1}\\ &= 6\sqrt{10\times 8}\times(-1)\\ &= -6\sqrt{80} = -24\sqrt{5}\end{aligned}$$

Explanation. — To determine the sign of the product, each imaginary number is reduced to the form $b\sqrt{-1}$. The numbers are then multiplied as ordinary radicals, subject to (A), § 214, that $\sqrt{-1}\times\sqrt{-1}=-1$.

24. Multiply $\sqrt{-2}+3\sqrt{-3}$ by $4\sqrt{-2}-\sqrt{-3}$.

First Solution

$\sqrt{-2}+3\sqrt{-3}=(\sqrt{2}+3\sqrt{3})\sqrt{-1}$,

$4\sqrt{-2}-\sqrt{-3}=(4\sqrt{2}-\sqrt{3})\sqrt{-1}$;

$\therefore (\sqrt{-2}+3\sqrt{-3})(4\sqrt{-2}-\sqrt{-3})$

$=(\sqrt{2}+3\sqrt{3})(4\sqrt{2}-\sqrt{3})(\sqrt{-1})^2$

$=(8+12\sqrt{6}-\sqrt{6}-9)(-1)=1-11\sqrt{6}$.

Second Solution

$$\begin{array}{r}\sqrt{-2}+3\sqrt{-3}\\ 4\sqrt{-2}-\sqrt{-3}\\ \hline -4\sqrt{4}-12\sqrt{6}\\ +3\sqrt{9}+\quad\sqrt{6}\\ \hline 1\quad -11\sqrt{6}\end{array}$$

Multiply:

25. $3\sqrt{-5}$ by $2\sqrt{-15}$.

26. $4\sqrt{-27}$ by $\sqrt{-12}$.

27. $2\sqrt{-8}$ by $5\sqrt{-3}$.

28. $8\sqrt{-1}$ by $\sqrt{-b^3}$.

29. $\sqrt{-125}$ by $\sqrt{-108}$.

30. $\sqrt{-100}$ by $\sqrt{-30}$.

31. $\sqrt{-6}+\sqrt{-3}$ by $\sqrt{-6}-\sqrt{-3}$.

Multiply:

32. $\sqrt{-ab}+\sqrt{-a}$ by $\sqrt{-ab}-\sqrt{-a}$.

33. $\sqrt{-xy}+\sqrt{-x}$ by $\sqrt{-xy}+\sqrt{-x}$.

34. $\sqrt{-50}-\sqrt{-12}$ by $\sqrt{-8}-\sqrt{-75}$.

35. $\sqrt{-a}+\sqrt{-b}+\sqrt{-c}$ by $\sqrt{-a}+\sqrt{-b}-\sqrt{-c}$.

36. Divide $\sqrt{-12}$ by $\sqrt{-3}$.

SOLUTION. $\dfrac{\sqrt{-12}}{\sqrt{-3}}=\dfrac{\sqrt{12}\sqrt{-1}}{\sqrt{3}\sqrt{-1}}=\dfrac{\sqrt{12}}{\sqrt{3}}=\sqrt{4}=2.$

37. Divide $\sqrt{12}$ by $\sqrt{-3}$.

SOLUTION

$$\frac{\sqrt{12}}{\sqrt{-3}}=\frac{\sqrt{12}}{\sqrt{3}\sqrt{-1}}=\frac{\sqrt{4}}{\sqrt{-1}}=\frac{2}{\sqrt{-1}}$$

$$=\frac{2\sqrt{-1}}{-1}=-2\sqrt{-1}.$$

38. Divide 5 by $(\sqrt{-1})^3$.

SOLUTION

$$\frac{5}{(\sqrt{-1})^3}=\frac{5(+1)}{(\sqrt{-1})^3}=\frac{5(\sqrt{-1})^4}{(\sqrt{-1})^3}=5\sqrt{-1}.$$

Divide:

39. $\sqrt{-18}$ by $\sqrt{-3}$.

40. $\sqrt{27}$ by $\sqrt{-3}$.

41. $14\sqrt{-5}$ by $2\sqrt{-7}$.

42. $-\sqrt{-a^2}$ by $\sqrt{-b^2}$.

43. 1 by $\sqrt{-1}$.

44. $\sqrt{8}+3\sqrt{14}$ by $\sqrt{-2}$.

45. $\sqrt{12}+\sqrt{3}$ by $\sqrt{-3}$.

46. -2 by $\sqrt{-1}$.

47. $(\sqrt{-1})^5$ by $\frac{1}{3}\sqrt{-1}$.

48. $(\sqrt{-1})^3$ by $(\sqrt{-1})^{15}$.

49. $\sqrt{4ab}$ by $\sqrt{-bc}$.

50. $(\sqrt{-1})^{14}$ by $-\frac{1}{2}\sqrt{-1}$.

51. $(\sqrt{-1})^{10}$ by $(\sqrt{-1})^{-2}$.

52. $\sqrt{-a}+b\sqrt{-1}$ by $\sqrt{-ab}$.

53. $\sqrt{-4}$ by $\sqrt{-2}\cdot\sqrt{-2}\cdot\sqrt{-1}$.

QUADRATIC EQUATIONS

217. Define and illustrate the following kinds of equations:

1. **Quadratic.**
2. **Second degree.**
3. **Pure quadratic.**
4. **Incomplete quadratic.**
5. **Affected quadratic.**
6. **Complete quadratic.**

PURE QUADRATIC EQUATIONS

218. Since pure quadratics contain only the second power of the unknown number, they may be reduced to the *general form* $ax^2 = c$, in which a represents the coefficient of x^2, and c the sum of the terms that do not involve x^2.

219. PRINCIPLE. — *Every pure quadratic equation has two roots, numerically equal but opposite in sign.*

It is proved in § 269 that *every* quadratic equation has two roots and only two roots.

EXERCISES

220. 1. Find the roots of the equation $3x^2 + 15 = 0$.

SOLUTION.	$3x^2 + 15 = 0.$
Transpose,	$3x^2 = -15.$
Divide by 3,	$x^2 = -5.$
Extract the square root, Ax. 7,	$x = \pm\sqrt{-5}.$

VERIFICATION. — The given equation becomes $0 = 0$, and is therefore satisfied when either $+\sqrt{-5}$ or $-\sqrt{-5}$ is substituted for x.

Solve, and verify each root:

2. $2x^2 - 4 = 4.$
3. $3x^2 + 2x^2 = 45.$
4. $12 + 3x^2 = 60.$
5. $12x^2 + 60 = 0.$
6. $3x^2 = 108.$
7. $4x^2 = \frac{1}{16}.$
8. $\frac{1}{4}x^2 = 8.$
9. $\frac{3}{4}x^2 + 18 = 30.$
10. $(x + 3)^2 = 6x + 6.$
11. $(x + 5)^2 = 10x + 41.$
12. $(x + 4)^2 = 8x + 24.$
13. $7x^2 - 25 = 5x^2 + 73.$

Solve and verify:

14. $4 + x^2 = 2(x + 12) - 2x.$

15. $(x + 2)^2 = 2x(x + 2) + 12.$

16. $(x - 3)^2 + 6(x - 1) = -9.$

17. $(x - 5)^2 - 10 = 5(7 - 2x).$

18. $(x + 2)^2 - 4(x + 2) - 2 = 2.$

19. $(x - 3)^2 + 10x = x(4 + 2x).$

20. $3(x^2 + 4) + 5x = 5(6 + x).$

21. $(x + 2)^2 - 4(x + 1) + 4 = 28.$

22. $4x(x + 2) - 5 = 12 - (x - 4)^2.$

23. $2(3 - 2x) + 20 = (x - 1)^2 - 2x.$

24. $\dfrac{x}{12} + \dfrac{x^2 - 15}{5x} = \dfrac{x}{5}.$

25. $\dfrac{x + 3}{x - 3} + \dfrac{x - 3}{x + 3} = 4.$

26. $\dfrac{x - 2}{x + 1} + \dfrac{x + 2}{x - 1} = -1.$

27. $\dfrac{x - 3}{x - 2} + \dfrac{x + 3}{x + 2} = 1\tfrac{7}{8}.$

28. $\dfrac{x - 2}{x + 2} - \dfrac{x + 2}{2 - x} = \dfrac{40}{x^2 - 4}.$

29. $\dfrac{x + 7}{x^2 - 7x} - \dfrac{x - 7}{x^2 + 7x} = \dfrac{7}{x^2 - 73}.$

30. What negative number is equal to its reciprocal?

31. If 25 is added to the square of a certain number, the sum is equal to the square of 13. What is the number?

32. When 5 is taken from a certain number, and also added to it, the product of these results is 75. Find the number.

33. A certain number multiplied by $\frac{1}{4}$ of itself is equal to 16. Find the number.

34. The area of a sheet of mica is 48 square inches and its length is $1\frac{1}{3}$ times its width. Find its length and its width.

35. At 75 cents per square yard, enough linoleum was purchased for \$36 to cover a rectangular floor whose length was 3 times its breadth. Find the dimensions of the floor.

36. The sum of the squares of two numbers is 394, and the difference of their squares is 56. What are the numbers?

37. A man had a rectangular field the width of which was $\frac{2}{3}$ of its length. He built a fence across it so that one of the two parts formed a square containing 10 acres. Find the dimensions of the original field in rods.

AFFECTED QUADRATIC EQUATIONS

221. Since affected quadratic equations contain both the second and the first powers of the unknown number, they may always be reduced to the general form $ax^2 + bx + c = 0$, in which a, b, and c may represent any numbers whatever, and x, the unknown number. The term c is called the **absolute term.**

222. Solution of affected quadratics by factoring.

Reduce the equation to the form $ax^2 + bx + c = 0$, factor the first member, and equate each factor to zero, as in § 82, thus obtaining two simple equations together equivalent to the given quadratic, subject to the exceptions given in § 108 as to equivalence.

EXERCISES

223. Solve by factoring, and verify results:

1. $x^2 + 7x + 12 = 0.$
2. $y^2 - 7y + 12 = 0.$
3. $x^2 + 4x = 21.$
4. $z^2 = z + 72.$
5. $y^2 = y + 110.$
6. $x^2 + 2x = 120.$
7. $y^2 - 20y = 96.$
8. $n^2 + 11n = -30.$
9. $36 = c^2 + 16c.$
10. $l^2 + 15l - 34 = 0.$
11. $r^2 = 6r + 135.$
12. $x^2 - 24 = 4(x + 2).$
13. $2x^2 + 3x - 2 = 0.$
14. $3x^2 + 11x - 4 = 0.$
15. $30 + r - r^2 = 0.$
16. $5x^2 + 9x = 2.$
17. $3x^2 - 7x - 6 = 0.$
18. $6x^2 - 5x = -1.$
19. $2x^2 + 15 = 3(2x + 5).$
20. $6(x^2 + 1) = 13x.$
21. $27y^2 - 3y - 14 = 0.$
22. $15s^2 - 4 = -17s.$
23. $9a^2 + 40 = 42a.$
24. $3(4x^2 + 2) + 25x = 8x.$
25. $2(3x^2 - 1) + 7x = 18x.$
26. $3 - 13x = 6(x^2 - 2).$
27. $4u(8u + 7) = 15.$
28. $4x(5x + 4) = 7x + 18.$

224. Solution of affected quadratics by completing the square. The general form of the perfect square of a binomial is

$$x^2 + 2\,ax + a^2.$$

Consequently, an expression like $x^2 + 2\,ax$ may be made a perfect square by adding the term a^2, which it will be observed is *the square of half the coefficient of x.*

This fact, as shown in the following solutions, is used to *complete the square* in one member of an *affected* quadratic, suitably prepared, so that it may be solved by *extracting the square root of both members* as was done in solving *pure* quadratics.

EXERCISES

225. 1. Solve the equation $x^2 - 3\,x - 10 = 0.$

SOLUTION. $$x^2 - 3\,x - 10 = 0.$$

Transpose the absolute term, $$x^2 - 3\,x = 10.$$

Complete the square in the first member by adding the square of half the coefficient of x, and add the same to the second member to preserve the equality,

$$x^2 - 3\,x + (\tfrac{3}{2})^2 = 10 + (\tfrac{3}{2})^2,$$

or $$x^2 - 3\,x + \tfrac{9}{4} = \tfrac{49}{4}.$$

Extract the square root of both members,

$$x - \tfrac{3}{2} = \pm\,\tfrac{7}{2};$$

whence, $$x = \tfrac{3}{2} + \tfrac{7}{2} \text{ or } \tfrac{3}{2} - \tfrac{7}{2};$$

that is, $$x = 5 \text{ or } -2.$$

VERIFICATION. — Either 5 or -2 substituted for x in the given equation reduces it to the identity $0 = 0$; that is, 5 and -2 are roots of the equation.

2. Solve the general quadratic equation $ax^2 + bx + c = 0$.

SOLUTION. $$ax^2 + bx + c = 0.$$

Transpose c and divide by a, $$x^2 + \frac{b}{a}x = -\frac{c}{a}.$$

Complete the square, etc., $$x^2 + \frac{b}{a}x + \frac{b^2}{4\,a^2} = \frac{b^2 - 4\,ac}{4\,a^2}.$$

Extract the square root, $$x + \frac{b}{2\,a} = \pm\,\frac{\sqrt{b^2 - 4\,ac}}{2\,a};$$

whence, $$x = \frac{-b \pm \sqrt{b^2 - 4\,ac}}{2\,a}.$$

Steps in the solution of an affected quadratic equation by the method of completing the square are:

1. *Transpose so that the terms containing* x^2 *and* x *are in one member and the known terms in the other.*

2. *Make the coefficient of* x^2 *positive unity by dividing both members by the coefficient of* x^2.

3. *Complete the square by adding to each member the square of half the coefficient of* x.

4. *Extract the square root of both members.*

5. *Solve the two simple equations thus obtained.*

Solve, and verify all results:

3. $x^2 - 2x = 143.$
4. $x^2 + 2x = 168.$
5. $x^2 - 4x = 117.$
6. $x^2 - 6x = 160.$
7. $8x = x^2 - 180.$
8. $x^2 + 2x = 120.$
9. $y^2 = 28y - 187.$
10. $x^2 - 12x = 189.$
11. $y^2 + 22y = -120.$
12. $z^2 - 180 = 3z.$
13. $v^2 + 15v = 54.$
14. $v^2 + 21v = -54.$
15. $2x^2 + 3x = 27.$
16. $3x^2 + 16x = 12.$
17. $2x^2 + 5x - 1 = 6.$
18. $4x^2 - 17x + 4 = 0.$
19. $6x^2 - 5x - 6 = 0.$
20. $.2x^2 + .9x = 3.5.$
21. $2x^2 - \frac{19}{2}x = \frac{5}{2}.$
22. $.03x^2 - .07x = .1.$

226. Solution of quadratics by the quadratic formula.

The general quadratic

$$ax^2 + bx + c = 0 \tag{1}$$

has been solved in exercise 2, § 225. Its roots are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{2}$$

Since (1) represents *any* quadratic equation, the student is now prepared to solve any quadratic equation whatever that contains one unknown number. The roots may be obtained by reducing it to the general form and employing (2) as a formula, known as the **quadratic formula.**

EXERCISES

227. 1. Solve the equation $6x^2 = x + 15$.

SOLUTION. — Writing the equation in the general form

$$6x^2 - x - 15 = 0,$$

we find that $a = 6$, $b = -1$, and $c = -15$.

$\therefore$ by (2), § 226,
$$x = \frac{1 \pm \sqrt{(-1)^2 - 4 \times 6(-15)}}{2 \times 6}$$
$$= \frac{1 \pm 19}{12} = \frac{5}{3} \text{ or } -\frac{3}{2}.$$

Solve by the quadratic formula:

2. $4x^2 - x - 3 = 0$.

3. $2x^2 + 5x + 2 = 0$.

4. $3x^2 + 11x + 6 = 0$.

5. $6x^2 + 2 = 7x$.

6. $5x^2 - 2x = 16$.

7. $4x^2 + 4x = 15$.

8. $2x^2 = 9 - 3x$.

9. $x(2x + 3) = -1$.

10. $13x = 3x^2 - 10$.

11. $7x^2 + 9x = 10$.

12. $5x^2 - 18x = 72$.

13. $1 - 3x = 2x^2$.

14. $3x^2 = 5x - 2$.

15. $4 = x(3x + 2)$.

16. $x^2 - 5x = -3$.

17. $3x^2 - 6x = -2$.

18. $4x^2 - 3x - 2 = 0$.

19. $x^2 + 10 = 6x$.

20. $x^2 = -4(x + 3)$.

21. $4(2x - 5) = x^2$.

22. $5x^2 + 18 = 6x$.

23. $x(3x + 4) = -2$.

228. General directions for solving quadratic equations.

1. *Reduce the equation to the general form* $ax^2 + bx + c = 0$.

2. *If the factors are readily seen, solve by factoring.*

3. *If the factors are not readily seen, solve by completing the square or by the formula.*

4. *Verify all results, reject roots introduced in the process of reducing the equation to the general form, and account for roots that have been removed.*

NOTE. — In general, no root is introduced by clearing an equation of fractions, provided that: fractions having a common denominator are combined; each fraction is expressed in its lowest terms; and both members are then multiplied by the lowest common denominator.

MISCELLANEOUS EXERCISES

229. Solve according to the general directions just given:

1. $2x^2 - 5x = 0.$
2. $x^2 - 30 = 13x.$
3. $r^2 + 27r = -140.$
4. $x^2 - 12x = 0.$
5. $18x^2 + 6x = 0.$
6. $5x^2 - 2x - 16 = 0.$
7. $8x^2 - 3 = -2x.$
8. $7x^2 + 2x = 32.$
9. $5x^2 = 4(x - 10).$
10. $x^2 - 4.3x = 27.3.$
11. $x^2 + .25x = .015.$
12. $\frac{1}{x+1} + \frac{3}{x-1} = \frac{10}{3}.$
13. $\frac{y}{4(y+1)} = \frac{y-3}{8}.$
14. $\frac{x^2}{9} + \frac{x^2 - 2x}{3x - 6} = \frac{35}{4}.$
15. $\frac{x^2}{x-2} - \frac{3x-5}{2} = \frac{x+2}{5}.$
16. $\frac{x^2}{4} - \frac{2x}{3} = 28.$
17. $\frac{4}{x^2 - 2x + 1} = \frac{1}{4}.$
18. $\frac{9x}{2x^2 + x} + \frac{3}{x-3} = 4.$
19. $\frac{1+x}{x-3} - \frac{x-1}{x-2} = \frac{4}{5}.$
20. $\frac{y^2}{y+3} = \frac{9}{y+3} + 8.$
21. $\frac{v+7}{v+5} + \frac{v+12}{v+6} = 7.$
22. $\frac{x-3}{x+4} + \frac{x+2}{x-2} = \frac{23}{10}.$
23. $\frac{2x+1}{1-2x} - \frac{5}{7} = \frac{x-8}{2}.$
24. $2 = \frac{(r+2)^2}{r^2 - 4} - \frac{r+5}{r-3}.$
25. $\frac{3x-1}{x-1} + \frac{2x+1}{x+1} = \frac{2x-4}{x-2}.$
26. $\frac{5y-2}{y^2 + 2y} + 5 = -\frac{7}{y^2 + 2y}.$

Find roots to the nearest hundredth:

27. $x^2 - 2x - 2 = 0.$
28. $z^2 + 2z - 1 = 0.$
29. $v^2 + 4v + .4 = 0.$
30. $y^2 + 4y + 2 = 0.$
31. $2s^2 + 4s - 7 = 0.$
32. $2x^2 - 5x + 1.2 = 0.$

LITERAL EQUATIONS

230. The methods of solution for literal quadratic equations are the same as for numerical quadratics. Results may be tested by substituting simple numerical values for the literal known numbers.

EXERCISES

231. Solve for x by the method best adapted, and verify:

1. $x^2 - b = 0.$
2. $6\,ax^2 - 54\,a^5 = 0.$
3. $x^2 - cd = cx - dx.$
4. $x^2 - 4\,bx - 12\,b^2 = 0.$
5. $x^2 + 2\,bx = b^2.$
6. $x^2 + 3\,ax = 10\,a^2.$
7. $x^2 - ax + bx + cx = 0.$
8. $(a-x)^2 = (3\,x+a)(x-a).$
9. $abx^2 + a^2x - b^2x = ab.$
10. $x^2 - 4\,bx - 7\,b^2 = 0.$
11. $ax^2 = (a-b)(a^2-b^2) - bx^2.$
12. $5\,cx - 2\,x^2 - 2\,c^2 = 0.$
13. $16\,x^2 + 3\,a^2 - 16\,ax = 0.$
14. $(c^2+1)x = cx^2 + c.$
15. $a^2x^2 + 2\,ax^2 = (a^2-1)^2 - x^2.$
16. $4\,x^2 + 12\,ax - 7\,a^2 = 0.$
17. $5\,x^2 - 10\,bx - 7\,b^2 = 0.$
18. $6\,ax^2 + abx = 2(6\,x + b).$
19. $x^2 - (b-a)c = x(a-b+c).$
20. $(b-c)x^2 + (c-a)x = b - a.$
21. $\dfrac{a}{x} + \dfrac{x}{a} = \dfrac{ab}{x}.$
22. $\dfrac{x}{a+b} - \dfrac{a-b}{x} = 0.$
23. $\dfrac{a^2x^2}{b^2} + \dfrac{b^2}{c^2} = \dfrac{2\,ax}{c}.$
24. $\dfrac{2\,x(b-x)}{3\,b-2\,x} = \dfrac{b}{4}.$
25. $\dfrac{a}{3} + \dfrac{5\,x}{4} - \dfrac{x^2}{3\,a} = 0.$
26. $\dfrac{x^2+1}{n^2x-2\,n} - \dfrac{1}{2-nx} = \dfrac{x}{n}.$
27. $\dfrac{x}{a+1} + \dfrac{2\,a-1}{x} = \dfrac{3\,ax}{x(a+1)}.$
28. $\dfrac{b}{x-a} + \dfrac{a}{x-b} - 2 = 0.$
29. $\dfrac{2\,x-a}{b} + 3 = \dfrac{4\,a}{2\,x-b}.$
30. $\dfrac{x+a}{x+b} + \dfrac{x-a}{x-b} = \dfrac{a^2+b^2}{x^2-b^2}.$
31. $\dfrac{bx}{a-x} + b = \dfrac{a(x+2\,b)}{a+b}.$
32. $\dfrac{1}{x-c} - \dfrac{1}{c} = \dfrac{1}{d} - \dfrac{1}{x-d}.$

RADICAL EQUATIONS

232. The student has learned how to free radical equations of radicals, the cases in §§ 209, 210, being such as lead to *simple equations.* The radical equations given here lead to *quadratic equations,* but the methods of freeing them of radicals are the same as in the cases already considered.

It was shown in § 211 that the processes of *rationalization* and *involution,* used in freeing radical equations of radicals, are likely to introduce roots that do not verify in accordance with the convention adopted, and in § 228 certain precautions against introducing roots by *clearing of fractions* were given.

It is important, therefore, to test the roots found in the solution of equations to see whether any are **extraneous**, as well as to examine the processes employed in reducing equations to see whether any roots have been removed (§ 109).

EXERCISES

233. Solve and verify, rejecting roots that do not satisfy the given equation, and accounting for roots that otherwise might be lost:

1. $2x - 3\sqrt{x} = 2.$
2. $x + 2\sqrt{x} = 3\sqrt{x}.$
3. $3x - \sqrt{x+3} - 1 = 0.$
4. $\sqrt{25-6x} + \sqrt{25+6x} = 8.$
5. $\sqrt{1-2x} - 2 = \sqrt{1-x}.$
6. $\sqrt{5-x} = \sqrt{3+x}\ \sqrt{x-5}.$
7. $\sqrt{x^2-b^2} = \sqrt{x+b}\ \sqrt{a+b}.$
8. $\sqrt{x+3} + \sqrt{4x+1} - \sqrt{10x+4} = 0.$

Find roots to the nearest hundredth:

9. $\sqrt{x^2+9} = \sqrt{2x+3}\ \sqrt{2x-3}.$
10. $\sqrt{2x+50} = \sqrt{x}\ \sqrt{x+2}.$
11. $\sqrt{x^2+3} = \sqrt{2x+1}\ \sqrt{2x-1}.$
12. $\sqrt{x+6} = \sqrt{3x} - 2.$
13. $\sqrt{2x} = \sqrt{x+2} + 1.$
14. $\sqrt{x^2+5} = \sqrt{2x+3}\ \sqrt{x-2}.$

Solve for x, and verify as directed on page 169:

15. $\sqrt{3x-5}+\sqrt{x-9}=\sqrt{4x-4}$.

16. $\dfrac{\sqrt{x+2a}-\sqrt{x-2a}}{\sqrt{x-2a}+\sqrt{x+2a}}=\dfrac{x}{2a}$.

17. $x+\sqrt{x^2+m^2}=\dfrac{2m^2}{\sqrt{x^2+m^2}}$.

18. $x+\sqrt{x^2-a^2}=\dfrac{a^2}{\sqrt{x^2-a^2}}$.

19. $\dfrac{2x+\sqrt{4x^2-1}}{2x-\sqrt{4x^2-1}}=4$.

20. $\sqrt{\dfrac{x-a}{x+a}}+\sqrt{\dfrac{x+a}{x-a}}=a^2$.

21. $\sqrt{x+a^2}-\sqrt{x-2a^2}=\sqrt{2x-5a^2}$.

22. $\sqrt{mn-x}-\sqrt{x}\sqrt{mn-1}=\sqrt{mn}\sqrt{1-x}$.

Problems

234. 1. The product of two numbers is 14 and their sum is 9. Find the numbers.

2. Separate 16 into two parts whose product is 48.

3. Separate 24 into two parts whose product is 128.

4. Find two consecutive integers whose product is 156.

5. The sum of the squares of two consecutive integers is 265. What are the numbers?

6. The difference between a certain number and its reciprocal is $\frac{63}{8}$. Find the number.

7. The sum of a certain number and its reciprocal is $\frac{25}{12}$. Find the number.

8. The sum of the reciprocals of two consecutive integers is $\frac{7}{12}$. Find the integers.

9. If a times the reciprocal of a number is added to the number, the result is $a+1$. What is the number?

10. The length of a sheet of paper is 14 inches more than its width and its area is 912 square inches. Find its length.

11. Find two consecutive even integers the sum of whose squares is $2(a^2+1)$.

12. A rectangular garden is 12 rods longer than it is wide and it contains 1 acre. What are its dimensions?

13. The area of a car floor is 306 square feet. If its length is 2 feet more than 4 times its width, what is its width?

14. The area of a tablet is 2838 square inches. If its length exceeds its width by 23 inches, what are its dimensions?

15. An ice bill for a month was $4.80. If the number of cakes used was 4 less than the number of cents paid per cake, how many cakes were used?

16. The height of a box is 5 feet less than its length and 2 inches more than its width. If the area of the bottom is $8\frac{2}{3}$ square feet, what are the dimensions of the box?

17. A roll of parchment was worth $24. If the number of skins it contained was 20 more than the number of cents each skin cost, how many skins were there in the roll?

18. The sum of the three dimensions of a block is 35 feet and its width and height are equal. The area of the top exceeds that of the end by 50 square feet. Find its dimensions.

19. The sum of the three dimensions of a box is 58 inches and its length and width are equal. The area of the bottom exceeds that of one end by 176 square inches. Find its height.

20. A bale of cotton contains 21 cubic feet. Its length is $4\frac{1}{2}$ feet, and its width is $\frac{1}{3}$ of a foot less than its thickness. Find its width; its thickness.

21. A man sold raisins for $480. If he had sold 2 tons more and had charged $20 less per ton, he would have received the same amount. How many tons of raisins did he sell?

22. If a beet-sugar factory in Colorado sliced 200 tons less of beets per day, it would take 1 day longer to slice 6000 tons of them. How many beets are sliced per day?

23. The senior class at a school had a banquet that cost \$ 75. If there had been 5 persons less, the share of each would have been \$ .50 more. How many persons were there in the class?

SUGGESTION. — Let x = the number of persons. Then, $\frac{75}{x}$ = the amount each paid and $\frac{75}{x-5}$, the amount each would have paid had there been 5 persons less. Hence, $\frac{75}{x-5} - \frac{75}{x} = \frac{1}{2}$.

24. A party of people agreed to pay \$ 12 for the use of a launch. As 2 of them failed to pay, the share of each of the others was 50 cents more. How many persons were there in the party?

25. A bricklayer and his helper in a certain day laid 1500 bricks. If they had laid 25 bricks more per hour and had worked 2 hours less time, they would have laid 1400 bricks. How many bricks did they lay per hour?

26. A rectangular park, 60 rods long and 40 rods wide, is surrounded by a street of uniform width, containing 1344 square rods. How wide is the street?

27. Two persons started at the same time and traveled toward a place 90 miles distant. A traveled 1 mile per hour faster than B, and reached the place 1 hour before him. At what rate did each travel?

28. If the rate of a sailing vessel was $1\frac{1}{4}$ knots more per hour, it would take $\frac{1}{2}$ of an hour less time to travel 150 knots. Find the rate of the vessel per hour.

29. A man rode 90 miles. If he had traveled $\frac{2}{11}$ of a mile more per hour, he would have made the journey in 10 minutes less time. How long did the journey last?

30. A picture that is 18 inches by 12 inches has a frame of uniform width whose area is equal to that of the picture. Find the width of the frame.

31. A tank can be filled by two pipes in $24\frac{3}{4}$ minutes. If it takes the smaller pipe 10 minutes longer to fill the tank than it does the larger pipe, in what time can the tank be filled by each pipe?

SUGGESTION. — Let $x =$ the number of minutes required by the larger pipe and $x + 10 =$ the number required by the smaller pipe.

Then, $\frac{1}{x} + \frac{1}{x+10} = \frac{1}{24\frac{3}{4}}$.

32. A tank can be filled by two pipes in 35 minutes. If the larger pipe alone can fill it in 24 minutes less time than the smaller pipe, in what time can each fill the tank?

33. A and B together can do a piece of work in 3 days. If it takes A working alone $1\frac{3}{4}$ days longer than it does B, in how many days can each do the work alone?

34. A cistern can be emptied by two pipes in $3\frac{1}{3}$ hours. The larger pipe alone can empty it in $1\frac{1}{2}$ hours less time than the smaller pipe. In what time can each pipe empty the cistern?

35. A farmer bought a horse for x dollars and sold it for $ 75, thus making a profit of x %. Find x.

36. A jeweler sold a clock for $ 24, thus gaining a per cent equal to the number of dollars the clock cost. How much did the clock cost?

37. If a man puts $ 2000 at interest, compounded annually, and at the end of 2 years finds that it amounts to $ 2121.80, what rate of interest is he receiving?

38. Find the price of eggs per dozen, when 2 less for 30 cents raises the price 2 cents per dozen.

39. By receiving two successive discounts, a dealer bought for $ 9 silverware that was listed at $ 20. What were the discounts in per cent, if the first was 5 times the second?

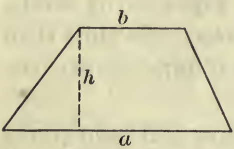
40. Each page of a book of 400 pages was 10 inches by 6 inches. In later editions, the publishers saved 1550 square inches of paper by cutting down the margin equally on every side. By what width was the margin reduced?

Formulæ

235. Solve the formula:

1. $S = 4\pi r^2$, for r.
2. $P = \dfrac{nd^2}{2.5}$, for d.
3. $F = \dfrac{1.5\,WS^2}{R}$, for S.
4. $V = \frac{1}{12}\pi d^2 h$, for d.
5. $b = \sqrt{\dfrac{4\,a^2r^2}{4\,r^2 - a^2}}$, for r.
6. $S = a + vt - 16t^2$, for t.
7. $h = r - \sqrt{r^2 - (\frac{1}{2}w)^2}$, for w.

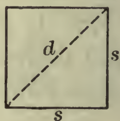
8. The formula $A = bh$ gives the area A of a parallelogram in terms of its base b and altitude h. If the area of a parallelogram is 96 square feet and its base is 4 feet more than twice its height, what is its height? its base?



9. The area A of a trapezoid is expressed by the formula $A = \frac{1}{2}h(a + b)$. If the lower base a of a trapezoid is 5 feet longer than the upper base b, the altitude h is 1 foot shorter than b, and the area is 92 square feet, what are its dimensions?

10. The square of the hypotenuse (h) of a right triangle is equal to the sum of the squares of the other two sides (a and b). Write the formula for h.

11. From the above formula and the figure, deduce a formula for the diagonal (d) of a square whose side is s.



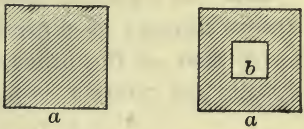
12. Find the diagonal of a square whose side is 8 feet.

13. If a baseball diamond is 90 feet square, what is the distance, to the nearest tenth of a foot, from first base to third base?

14. Write the formula for the diagonal of a rectangle whose length is a and width is b. Solve for a.

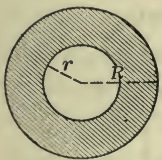
15. The diagonal of a rectangle is 10 feet long. The rectangle is 2 feet longer than it is wide. Find its dimensions.

16. Express by an equality the area (A) of a square whose side is a; the area (A) of a hollow square a units on the outside and b units on the inside.



17. The area of a hollow square is 40 square inches. If the outside dimension is twice the inside dimension plus 1 inch, what is the inside dimension?

18. The area of a flat ring is the difference between the areas of two circles of radii R and r, respectively, or $A = \pi(R^2 - r^2)$. Solve for r.

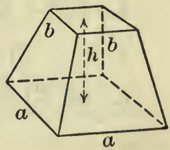


19. When the area of a ring is 1320 square feet and $R = r + 2$, what is the value of r? (Use $\pi = 3\frac{1}{7}$.)

20. The pressure P of the wind against a surface, in pounds per square foot, is computed from $P = .005\,V^2$, in which V is the velocity of the wind in miles per hour. Solve for V.

21. What is the velocity of the wind when it exerts a total pressure of 2.7 tons on a sign board 30 feet by 10 feet?

22. The formula for the volume of the frustum of a square pyramid is $V = \frac{1}{3}h(a^2 + ab + b^2)$. Find a and b when $V = 98$, $h = 6$, and $a = b + 2$.



23. If L is the length of a pendulum that oscillates once in T seconds, and l the length of one that oscillates once in t seconds, then $\frac{L}{l} = \frac{T^2}{t^2}$. Solve for t.

24. If a pendulum 39.1 inches long oscillates once per second, how often does a pendulum 351.9 inches long oscillate?

25. A line is said to be divided in extreme and mean ratio when the longer part is a mean proportional between the whole line and the shorter part. Write the proportion for a line a whose longer part is b and shorter part c. Solve for b.

26. Divide a line $4\frac{1}{2}$ feet long in extreme and mean ratio.

EQUATIONS IN THE QUADRATIC FORM

236. An equation that contains but two powers of an unknown number or expression, the exponent of one power being twice that of the other, as $ax^{2n}+bx^n+c=0$, in which n represents any number, is in the **quadratic form.**

EXERCISES

237. Solve the following equations:

1. $x^4+x^2-20=0$.

SOLUTION

$$x^4+x^2-20=0.$$
$$(x^2-4)(x^2+5)=0.$$
$$\therefore x^2-4=0 \text{ or } x^2+5=0,$$

and $\quad x=\pm 2$ or $\pm\sqrt{-5}$.

Any one of these values substituted for x in the given equation satisfies the equation, and is therefore a root of it.

7. $x^{\frac{1}{2}}-x^{\frac{1}{4}}=6$.

SOLUTION

$$x^{\frac{1}{2}}-x^{\frac{1}{4}}=6.$$
$$x^{\frac{1}{2}}-x^{\frac{1}{4}}+(\tfrac{1}{2})^2=\tfrac{25}{4}.$$
$$x^{\frac{1}{4}}-\tfrac{1}{2}=\pm\tfrac{5}{2}.$$
$$\therefore x^{\frac{1}{4}}=3 \text{ or } -2;$$

whence, $\quad x=81$ or 16.

Since $x=16$ does not verify, 16 is not a root and should be rejected.

2. $x^4+17x^2-84=0$.
3. $3x^4+5x^2-8=0$.
4. $5x^4+6x^2-11=0$.
5. $(x-2)^2+3(x-2)=10$.
6. $(x^2+1)^2+4(x^2+1)=45$.
8. $x^{\frac{1}{3}}-3x^{\frac{1}{6}}=-2$.
9. $x^{\frac{1}{2}}+3x^{\frac{1}{4}}-28=0$.
10. $x+3\sqrt{x}=4$.
11. $x^{\frac{2}{3}}-4x^{\frac{1}{3}}=12$.
12. $x^{\frac{4}{3}}=17x^{\frac{2}{3}}-16$.
13. $x-4x^{\frac{2}{3}}+3x^{\frac{1}{3}}=0$.

SOLUTION. — Factor, $\quad x^{\frac{1}{3}}(x^{\frac{1}{3}}-1)(x^{\frac{1}{3}}-3)=0$;

that is, $\quad x^{\frac{1}{3}}=0,\ x^{\frac{1}{3}}-1=0$, or $x^{\frac{1}{3}}-3=0$;

whence, $\quad x^{\frac{1}{3}}=0$, 1, or 3.

Raise to the third power, $\quad x=0$, 1, or 27.

Each of these values of x satisfies the given equation and is a root of it.

14. $x^{\frac{3}{2}}-4x-5x^{\frac{1}{2}}=0$.
15. $x-3x^{\frac{3}{4}}+2x^{\frac{1}{2}}=0$.
16. $x+2x^{\frac{5}{3}}-3x^{\frac{4}{3}}=0$.
17. $5x=x\sqrt{x}+6\sqrt{x}$.
18. $3x=x\sqrt[3]{x}+2\sqrt[3]{x^2}$.
19. $2x+\sqrt{x}=15x\sqrt{x}$.

20. Solve $x^2 - 3x + 2\sqrt{x^2 - 3x + 6} = 18$.

Solution. — Adding 6 to both members, we have

$$x^2 - 3x + 6 + 2\sqrt{x^2 - 3x + 6} = 24. \quad (1)$$

Put p for $\sqrt{x^2 - 3x + 6}$ and p^2 for $x^2 - 3x + 6$.

Then, $$p^2 + 2p = 24. \quad (2)$$

Solving, we have $$p = 4 \text{ or } -6; \quad (3)$$

that is, $$\sqrt{x^2 - 3x + 6} = 4, \quad (4)$$

or $$\sqrt{x^2 - 3x + 6} = -6. \quad (5)$$

Square (4), $$x^2 - 3x + 6 = 16. \quad (6)$$

Solving (6), we have $$x = 5 \text{ or } -2.$$

Since, in accordance with § 211, the radical in (5) cannot equal a negative number, $\sqrt{x^2 - 3x + 6} = -6$ is an impossible equation.

Hence, the only roots of the given equation are 5 and -2.

Solve and verify results:

21. $x - 2\sqrt{x + 1} = 7$.

22. $x^2 - x + 4\sqrt{x^2 - x - 8} = 20$.

23. Solve the equation $x^6 - 9x^3 + 8 = 0$.

Solution. $$x^6 - 9x^3 + 8 = 0. \quad (1)$$

Factor, $$(x^3 - 1)(x^3 - 8) = 0. \quad (2)$$

Therefore, $$x^3 - 1 = 0, \quad (3)$$

or $$x^3 - 8 = 0. \quad (4)$$

If the values of x are found by transposing the known terms in (3) and (4) and then extracting the cube root of each member, only *one* value of x will be obtained from each equation. But if the equations are factored, *three* values of x are obtained for each.

Factor (3), $$(x - 1)(x^2 + x + 1) = 0, \quad (5)$$

and (4), $$(x - 2)(x^2 + 2x + 4) = 0. \quad (6)$$

Writing each factor equal to zero, and solving, we have:

From (5), $x = 1,\ \frac{1}{2}(-1 + \sqrt{-3}),\ \frac{1}{2}(-1 - \sqrt{-3})$. (7)

From (6), $x = 2,\ -1 + \sqrt{-3},\ -1 - \sqrt{-3}$. (8)

Note. — Since the values of x in (7) are obtained by factoring $x^3 - 1 = 0$, they may be regarded as the *three cube roots of the number* 1.

Also, the values of x in (8) may be regarded as the *three cube roots of the number* 8 (§ 166).

Find the three cube roots of:

24. 27. **25.** -27. **26.** 64. **27.** 125. **28.** -64.

Solve:

29. $x^4 - 81 = 0$. **30.** $x^6 - 64 = 0$.

31. $x^4 + 4x^3 - 8x + 3 = 0$.

SOLUTION

$$x^4 + 4x^3 - 8x + 3 = 0.$$

Factor, § 75, $(x-1)(x+3)(x^2+2x-1) = 0$;

whence, $x = 1, -3, -1 \pm \sqrt{2}$.

32. $x^4 + 2x^3 - x = 30$. **34.** $x^4 + 2x^3 + 5x^2 + 4x = 60$.

33. $x^4 - 2x^3 + x = 132$. **35.** $x^4 - 6x^3 + 15x^2 - 18x = -8$.

36. $\dfrac{x^2}{x+1} + \dfrac{x+1}{x^2} = \dfrac{25}{12}$.

SUGGESTION. — Since the second term is the *reciprocal* of the first, put p for the first term and $\frac{1}{p}$ for the second.

Then, $p + \dfrac{1}{p} = \dfrac{25}{12}$.

37. $\dfrac{x^2+x}{2} + \dfrac{2}{x^2+x} = 2$. **38.** $\dfrac{x^2+1}{4} + \dfrac{4}{x^2+1} = \dfrac{5}{2}$.

MISCELLANEOUS EXERCISES

238. Solve the following equations:

1. $\sqrt[4]{x} + 3\sqrt{x} = 30$. **4.** $x = 11 - 3\sqrt{x+7}$.

2. $ax^{2n} + bx^n + c = 0$. **5.** $x^6 + 9x^3 + 8 = 0$.

3. $x - 7x^{\frac{2}{3}} + 10x^{\frac{1}{3}} = 0$. **6.** $x^{-\frac{2}{3}} - 5x^{-\frac{1}{3}} + 4 = 0$.

7. $x^2 - 5x + 5\sqrt{x^2 - 5x + 1} = 49$.

8. $(x^2 - x)^2 - (x^2 - x) - 132 = 0$.

9. $x^2 + x + 1 - \dfrac{1}{x^2+x+1} = \dfrac{8}{3}$.

10. $x^2 - 2x + \dfrac{4}{x^2 - 2x + 1} = 4$. **12.** $\left(x + \dfrac{1}{x}\right)^2 - 2\left(x + \dfrac{1}{x}\right) = \dfrac{5}{4}$.

11. $\dfrac{x}{x^2-1} + \dfrac{x^2-1}{x} = -\dfrac{13}{6}$. **13.** $\left(\dfrac{1+x^2}{x}\right)^2 + 2\left(\dfrac{1+x^2}{x}\right) = 8$.

SIMULTANEOUS EQUATIONS INVOLVING QUADRATICS

239. Two simultaneous *quadratic* equations in two unknown numbers generally lead to equations of the *fourth degree*, and they cannot be solved usually by quadratic methods, but some simultaneous equations *involving* quadratics are solvable by quadratic methods, as in the following cases.

240. When one equation is simple and the other of higher degree.

Equations of this class may be solved by substitution.

EXERCISES

241. 1. Solve the equations $\begin{cases} x + y = 5, & (1) \\ x^2 + 2y^2 = 17. & (2) \end{cases}$

SOLUTION. — From (1), $y = 5 - x.$ (3)

Substitute (3) in (2), $x^2 + 2(5 - x)^2 = 17.$ (4)

Solving (4), we have $x = 3$ or $\frac{11}{3}$. (5)

Substitute 3 for x in (3), $y = 2.$ (6)

Substitute $\frac{11}{3}$ for x in (3), $y = \frac{4}{3}.$ (7)

Hence, x and y each have two corresponding values associated as follows:

$$\begin{cases} x = 3;\ \frac{11}{3}; \\ y = 2;\ \frac{4}{3}. \end{cases}$$

Solve the following equations:

2. $\begin{cases} x = 3y, \\ x^2 + y^2 = 40. \end{cases}$

3. $\begin{cases} x + y = 3, \\ x^2 + 2xy = 8. \end{cases}$

4. $\begin{cases} x^2 - 2y^2 = 7, \\ x - y = 8. \end{cases}$

5. $\begin{cases} 3y^2 - z^2 = 8, \\ 2y = 2 - z. \end{cases}$

6. $\begin{cases} x^3 + y^3 = 9, \\ x + y = 3. \end{cases}$

7. $\begin{cases} 2y(x - 2) = 7, \\ 2x = 3y. \end{cases}$

242. An equation that is not affected by interchanging the unknown numbers involved is called a **symmetrical equation.**

$x^2 + xy + y^2 = 7$ and $3x^2 + 3y^2 = 4$ are symmetrical equations.

243. When both equations are symmetrical.

Though equations of this class may be solved by substitution, it is better to solve first for $x+y$ and $x-y$ and then for x and y.

EXERCISES

244. 1. Solve the equations $\begin{cases} x+y=11, & (1) \\ xy=30. & (2) \end{cases}$

SOLUTION. — Square (1), $x^2+2xy+y^2=121.$ (3)

Multiply (2) by 4, $4xy=120.$ (4)

Subtract (4) from (3), $x^2-2xy+y^2=1.$ (5)

Extract the square root, $x-y=\pm 1.$ (6)

From (1) + (6), $x=6 \text{ or } 5.$

From (1) − (6), $y=5 \text{ or } 6.$

2. Solve the equations $\begin{cases} x^2+y^2=13, & (1) \\ x+y=5. & (2) \end{cases}$

SUGGESTION. — From the square of (2) subtract (1); then subtract this result from (1) and proceed as in exercise 1.

3. Solve the equations $\begin{cases} x^4+y^4=97, & (1) \\ x+y=1. & (2) \end{cases}$

SOLUTION. — Raising (2) to the fourth power, we have

$$x^4+4x^3y+6x^2y^2+4xy^3+y^4=1. \quad (3)$$

Subtract (1) from (3), $4x^3y+6x^2y^2+4xy^3=-96.$ (4)

Divide (4) by 2, $2x^3y+3x^2y^2+2xy^3=-48.$ (5)

$2xy \times$ square of (2), $2x^3y+4x^2y^2+2xy^3=2xy.$ (6)

Subtract (5) from (6), $x^2y^2-2xy=48.$ (7)

Solve for xy, $xy=-6 \text{ or } 8.$ (8)

Equations (2) and (8) give two pairs of simultaneous equations,

$$\begin{cases} x+y=1 \\ xy=-6 \end{cases} \text{and} \begin{cases} x+y=1 \\ xy=8 \end{cases}$$

Solve as in exercise 1. The corresponding values of x and y are:

$$\begin{cases} x=3;\ -2; & \tfrac{1}{2}(1+\sqrt{-31}); & \tfrac{1}{2}(1-\sqrt{-31}); \\ y=-2;\ 3; & \tfrac{1}{2}(1-\sqrt{-31}); & \tfrac{1}{2}(1+\sqrt{-31}). \end{cases}$$

Solve the following equations:

4. $\begin{cases} x + y = 7, \\ xy = 6. \end{cases}$

5. $\begin{cases} x^2 + y^2 = 17, \\ xy = 4. \end{cases}$

6. $\begin{cases} x + y = 4, \\ x^2 + xy + y^2 = 13. \end{cases}$

7. $\begin{cases} x + y = 3, \\ x^3 + y^3 = 117. \end{cases}$

8. $\begin{cases} x^2 + xy + y^2 = 57, \\ x^2 + y^2 = 50. \end{cases}$

9. $\begin{cases} x^2 + y^2 = 26, \\ x^2 - xy + y^2 = 21. \end{cases}$

10. $\begin{cases} x^2 + y^2 = 13, \\ x + y + xy = 11. \end{cases}$

11. $\begin{cases} x + xy + y = 19, \\ x^2y^2 = 144. \end{cases}$

12. $\begin{cases} x^4 + y^4 = 17, \\ x + y = 3. \end{cases}$

13. $\begin{cases} x^4 + x^2y^2 + y^4 = 21, \\ x^2 + xy + y^2 = 7. \end{cases}$

245. An equation *all* of whose terms are of the same degree with respect to the unknown numbers is called a **homogeneous equation.**

$3x^2 + xy = y^2$ and $x^3 - 2y^3 = 0$ are homogeneous equations.

An equation like $2x^2 + xy + y^2 = 39$ is said to be **homogeneous in the unknown terms.**

246. When both equations are quadratic, one being homogeneous.

In this case elimination may always be effected by substitution, for by dividing the homogeneous equation through by y^2, it becomes a quadratic in $\frac{x}{y}$. The two values of $\frac{x}{y}$ obtained from this equation give two *simple* equations in x and y, each of which may be combined with the remaining quadratic equation as in §§ 240, 241.

Thus, $ax^2 + bxy + cy^2 = 0$ is the general form of the homogeneous equation in which a, b, and c are known numbers.

Dividing by y^2, we have $a\left(\frac{x}{y}\right)^2 + b\left(\frac{x}{y}\right) + c = 0$, a quadratic in $\frac{x}{y}$.

EXERCISES

247. 1. Solve the equations $\begin{cases} x^2 + 3x - y = 5, & (1) \\ 5x^2 + 4xy - y^2 = 0. & (2) \end{cases}$

Solution. — Dividing (2) by y^2 gives $5\left(\frac{x}{y}\right)^2 + 4\left(\frac{x}{y}\right) - 1 = 0$, a quadratic in $\frac{x}{y}$ which may be solved by factoring or by completing the square.

To avoid fractions, however, (2) may be factored at once; thus,

$$(x + y)(5x - y) = 0.$$

$$\therefore y = -x \text{ or } 5x.$$

Substituting $-x$ for y in (1), simplifying, etc., we have

$$x^2 + 4x = 5.$$

Solving gives $$x = 1 \text{ or } -5. \quad (3)$$

$$\therefore y = -x = -1 \text{ or } 5. \quad (4)$$

Substituting $5x$ for y in (1), simplifying, etc., we have

$$x^2 - 2x = 5.$$

Solving gives $$x = 1 + \sqrt{6} \text{ or } 1 - \sqrt{6}. \quad (5)$$

$$\therefore y = 5x = 5(1 + \sqrt{6}) \text{ or } 5(1 - \sqrt{6}). \quad (6)$$

Hence, from (3), (4), (5), and (6) the roots of the given equations are

$$\begin{cases} x = 1; & -5; & 1 + \sqrt{6}; & 1 - \sqrt{6}; \\ y = -1; & 5; & 5(1 + \sqrt{6}); & 5(1 - \sqrt{6}). \end{cases}$$

Solve the following equations:

2. $\begin{cases} 2x^2 - 3y - y^2 = 8, \\ 6x^2 - 5xy - 6y^2 = 0. \end{cases}$

3. $\begin{cases} 5x^2 + 8xy - 4y^2 = 0, \\ xy + 2y^2 = 60. \end{cases}$

4. $\begin{cases} 2x^2 - xy - y^2 = 0, \\ 4x^2 + 4xy + y^2 = 36. \end{cases}$

5. $\begin{cases} 6x^2 + xy - 12y^2 = 0, \\ x^2 + xy - y = 1. \end{cases}$

6. $\begin{cases} 3x^2 - 7xy - 40y^2 = 0, \\ x^2 - xy - 12y^2 = 8. \end{cases}$

7. $\begin{cases} x^2 - xy - y^2 = 20, \\ 3x^2 - 13xy + 12y^2 = 0. \end{cases}$

8. $\begin{cases} 3x^2 - 7xy + 4y^2 = 0, \\ 5x^2 - 7xy + 3y^2 = 4. \end{cases}$

9. $\begin{cases} x^2 + y^2 + x - y = 12, \\ 3x^2 + 2xy - y^2 = 0. \end{cases}$

248. When both equations are quadratic and homogeneous in the unknown terms.

In this case either:

Substitute vy for x, solve for y^2 in each equation, and compare the values of y^2 thus found, forming a quadratic in v.

Or, eliminate the absolute term, forming a homogeneous equation; then proceed as in §§ 246, 247.

EXERCISES

249. 1. Solve the equations $\begin{cases} x^2 - xy + y^2 = 21, & (1) \\ y^2 - 2xy = -15. & (2) \end{cases}$

FIRST SOLUTION. — Assume $x = vy$. (3)

Substitute (3) in (1), $v^2y^2 - vy^2 + y^2 = 21$. (4)

Substitute (3) in (2), $y^2 - 2vy^2 = -15$. (5)

Solve (4) for y^2, $y^2 = \dfrac{21}{v^2 - v + 1}$. (6)

Solve (5) for y^2, $y^2 = \dfrac{15}{2v - 1}$. (7)

Compare the values of y^2, $\dfrac{15}{2v-1} = \dfrac{21}{v^2 - v + 1}$. (8)

Clear, etc., $5v^2 - 19v + 12 = 0$. (9)

Factor, $(v - 3)(5v - 4) = 0$. (10)

$\therefore v = 3 \text{ or } \frac{4}{5}$. (11)

Substitute 3 for v in (7) or in (6), $y = \pm\sqrt{3}$,
and since $x = vy$, $x = \pm 3\sqrt{3}$. (12)

Substitute $\frac{4}{5}$ for v in (7) or in (6), $y = \pm 5$,
and since $x = vy$, $x = \pm 4$. (13)

When the double sign is used, as in (12) and in (13), it is understood that the roots shall be associated by taking the *upper* signs together and the *lower* signs together.

Hence, $\begin{cases} x = 3\sqrt{3}; & -3\sqrt{3}; & 4; & -4; \\ y = \sqrt{3}; & -\sqrt{3}; & 5; & -5. \end{cases}$

SUGGESTION FOR SECOND SOLUTION. — Multiplying (1) by 5 and (2) by 7, and adding the results, we eliminate the absolute term and obtain the homogeneous equation $5x^2 - 19xy + 12y^2 = 0$, which may be solved with either of the given equations, as in § 246.

Solve the following equations:

2. $\begin{cases} 2x^2 - y^2 = 7, \\ y^2 - xy = -1. \end{cases}$

3. $\begin{cases} x^2 + xy = 24, \\ xy + 2y^2 = 16. \end{cases}$

4. $\begin{cases} x(x-y) = 6, \\ x^2 - 3y^2 = 3. \end{cases}$

5. $\begin{cases} x^2 + y^2 = 13, \\ xy + y^2 = 15 \end{cases}$

6. $\begin{cases} x^2 - xy = 15, \\ 2xy - y^2 = 16. \end{cases}$

7. $\begin{cases} x^2 - xy - y^2 = 20, \\ x^2 - 3xy + 2y^2 = 8. \end{cases}$

8. $\begin{cases} 2x^2 - 3xy + 2y^2 = 100, \\ x^2 - y^2 = 75. \end{cases}$

9. $\begin{cases} x^2 - 5xy + 3y^2 = 8, \\ 3x^2 + xy + y^2 = 24. \end{cases}$

250. Special devices.

Many systems of equations belonging to the preceding classes and others not included in them may be solved readily by *special devices*, as illustrated in the following exercises. Though it is impossible to lay down any fixed line of procedure, the object often aimed at is to find values for *any two* of the expressions $x + y$, $x - y$, and xy from which the values of x and y may be obtained.

EXERCISES

251. 1. Solve the equations $\begin{cases} x^2 + xy = 20, & (1) \\ xy + y^2 = 5. & (2) \end{cases}$

SOLUTION

Add (1) and (2), $x^2 + 2xy + y^2 = 25.$ (3)

Extract the square root, $x + y = +5 \text{ or } -5.$ (4)

Subtract (2) from (1), $x^2 - y^2 = 15.$ (5)

Divide (5) by (4), $x - y = +3 \text{ or } -3.$ (6)

Add (4) and (6), etc., $x = 4 \text{ or } -4.$

Subtract (6) from (4), etc., $y = 1 \text{ or } -1.$

NOTE. — The first value of $x - y$ corresponds only to the first value of $x + y$, and the second value only to the second value.

Consequently, there are only two pairs of values of x and y.

2. Solve the equations $\begin{cases} x^2+y^2+x+y=14, \\ xy=3. \end{cases}$

Suggestion. — Adding twice the second equation to the first, we have
$$x^2+2\,xy+y^2+x+y=20, \text{ or } (x+y)^2+(x+y)=20,$$
which may be solved for $x+y$ and the results combined with $xy=3$.

Symmetrical except as to sign. — Whether both equations are symmetrical, or one is symmetrical and the other would be so if some of its signs were changed, or both are of the latter type, the method of solution is the same as in § 243.

3. Solve the equations $\begin{cases} x^2+y^2=53, & (1) \\ x-y=5. & (2) \end{cases}$

Suggestion. — Subtract the square of (2) from (1), obtaining $2\,xy=28$; add this equation to (1), and solve for $x+y$.

4. Solve the equations $\begin{cases} \dfrac{1}{x^2}+\dfrac{1}{y^2}=74, \\ \dfrac{1}{x}-\dfrac{1}{y}=2. \end{cases}$

Suggestion. — Proceed as in exercise 3, solving for $\frac{1}{x}+\frac{1}{y}$, then for $\frac{1}{x}$ and $\frac{1}{y}$, and finally for x and y.

Whether the equations are symmetrical or symmetrical except for the sign, it is often advantageous to substitute $u+v$ for x and $u-v$ for y.

5. Solve the equations $\begin{cases} x^4+y^4=82, & (1) \\ x-y=2. & (2) \end{cases}$

Solution. — Assume $$x=u+v, \quad (3)$$
and $$y=u-v. \quad (4)$$

Substitute these values in (1),
$$u^4+4\,u^3v+6\,u^2v^2+4\,uv^3+v^4$$
$$+\,u^4-4\,u^3v+6\,u^2v^2-4\,uv^3+v^4=82, \quad (5)$$
and in (2), $$2\,v=2. \quad (6)$$
Divide (5) by 2, $$u^4+6\,u^2v^2+v^4=41. \quad (7)$$
Divide (6) by 2, $$v=1. \quad (8)$$
Substitute 1 for v in (7) and solve, $$u=\pm 2 \text{ or } \pm\sqrt{-10}. \quad (9)$$

Hence, substituting (8) and (9) in (3) and (4), we find the corresponding values of x and y to be

$$\begin{cases} x = 3; & -1; & 1+\sqrt{-10}; & 1-\sqrt{-10}; \\ y = 1; & -3; & -1+\sqrt{-10}; & -1-\sqrt{-10}. \end{cases}$$

NOTE. — The given system of equations may be solved also by the method of exercise 3, § 244.

Division of one equation by the other. — The reduction of equations of higher degree to quadratics is often effected by dividing one of the given equations by the other, *member by member*.

6. Solve the equations $\begin{cases} x^4 + x^2y^2 + y^4 = 91, & (1) \\ x^2 - xy + y^2 = 7. & (2) \end{cases}$

SOLUTION. — Divide (1) by (2), $x^2 + xy + y^2 = 13.$ (3)

Subtract (2) from (3), $2xy = 6;$

whence, $xy = 3.$ (4)

Add (4) and (3), $x^2 + 2xy + y^2 = 16.$ (5)

Subtract (4) from (2), $x^2 - 2xy + y^2 = 4.$ (6)

Extract the square root of (5), $x + y = 4 \text{ or } -4.$ (7)

Extract the square root of (6), $x - y = 2 \text{ or } -2.$ (8)

Solving these simultaneous equations in (7) and (8), we have

$$\begin{cases} x = 3;\ 1;\ -1;\ -3; \\ y = 1;\ 3;\ -3;\ -1. \end{cases}$$

NOTE. — Since (7) and (8) have been derived independently, with the first value of $x + y$ we associate *each* value of $x - y$ in succession, and with the second value of $x + y$ each value of $x - y$ in succession, *in the same order. Consequently, there are four pairs of values of x and y.*

7. Solve the equations $\begin{cases} x^3 - y^3 = 54, & (1) \\ x - y = 6. & (2) \end{cases}$

SUGGESTION. — Divide (1) by (2) and solve the system made up of this result and (2).

Elimination of similar terms. — When the equations are quadratic and each is homogeneous except for one term, if these excepted terms are similar in the two equations, they may be eliminated and the solution of the system be made to depend on the case of § 246.

Some equations belonging to this class, namely, those that are homogeneous except for the *absolute term*, have been treated in § 248.

8. Solve the equations $\begin{cases} x^2 + 2xy = \frac{5}{2}y, \\ 2x^2 - xy + y^2 = 2y. \end{cases}$

SUGGESTION. — Eliminate the terms containing y and proceed as in § 246.

Using the methods illustrated in exercises 1–8, solve:

9. $\begin{cases} x^2 + xy = 30, \\ xy + y^2 = 6. \end{cases}$

10. $\begin{cases} p^2 + q^2 + p + q = 36, \\ pq = -15. \end{cases}$

11. $\begin{cases} y^2 + z^2 = 41, \\ y - z = 1. \end{cases}$

12. $\begin{cases} \dfrac{1}{x^2} + \dfrac{1}{y^2} = 13, \\ \dfrac{1}{x} - \dfrac{1}{y} = 1. \end{cases}$

13. $\begin{cases} x^4 + y^4 = 17, \\ x - y = 1. \end{cases}$

14. $\begin{cases} c^4 + c^2d^2 + d^4 = 3, \\ c^2 - cd + d^2 = 3. \end{cases}$

15. $\begin{cases} x^3 - y^3 = 64, \\ x - y = 4. \end{cases}$

16. $\begin{cases} x^2 + 2xy = 7y, \\ 2x^2 - xy + y^2 = 8y. \end{cases}$

MISCELLANEOUS EXERCISES

252. Solve the following systems of equations:

1. $\begin{cases} x - y = 4, \\ xy = -4. \end{cases}$

2. $\begin{cases} 3x^2 - 2y^2 = 19, \\ 2x^2 - 3y^2 = 6. \end{cases}$

3. $\begin{cases} x^2 + y^2 = 52, \\ 3x = 2y. \end{cases}$

4. $\begin{cases} x^4 + y^4 = 82, \\ x + y = 4. \end{cases}$

5. $\begin{cases} x^2 + xy = 77, \\ xy - y^2 = 12. \end{cases}$

6. $\begin{cases} 2x - y = 2, \\ 2x^2 + y^2 = \frac{3}{2}. \end{cases}$

7. $\begin{cases} x^2 + 3xy = y^2 + 23, \\ x + 3y = 9. \end{cases}$

8. $\begin{cases} x^2 + 4x + 3y = -1, \\ 2x^2 + 5xy + 2y^2 = 0. \end{cases}$

9. $\begin{cases} x^2 + 3xy - y^2 = 43, \\ x + 2y = 10. \end{cases}$

10. $\begin{cases} 2x^2 + 3xy + y^2 = 20, \\ 5x^2 + 4y^2 = 41. \end{cases}$

11. $\begin{cases} 2xy - y^2 = 12, \\ 3xy + 5x^2 = 104. \end{cases}$

12. $\begin{cases} x^2 + xy + y^2 = 84, \\ x - \sqrt{xy} + y = 6. \end{cases}$

Solve the following systems of equations:

13. $\begin{cases} 4x^2 - 2xy + y^2 = 13, \\ 8x^3 + y^3 = 65. \end{cases}$

14. $\begin{cases} 6x^2 + 6y^2 = 13xy, \\ x^2 - y^2 = 20. \end{cases}$

15. $\begin{cases} x^4 - y^4 = 175, \\ x^2 - y^2 = 7. \end{cases}$

16. $\begin{cases} x + y = 10, \\ \sqrt{x} + \sqrt{y} = 4. \end{cases}$

17. $\begin{cases} x^3 + y^3 = 225y, \\ x^2 - y^2 = 75. \end{cases}$

18. $\begin{cases} x^2 + y^2 - 3(x + y) = 8, \\ x + y + xy = 11. \end{cases}$

19. $\begin{cases} x^2 + y^2 = 3xy + 5, \\ x^4 + y^4 = 2. \end{cases}$

20. $\begin{cases} x^2 - 7xy + 12y^2 = 0, \\ xy + 3y = 2x + 21. \end{cases}$

21. $\begin{cases} (x + y)(x^2 + y^2) = 65, \\ (x - y)(x^2 - y^2) = 5. \end{cases}$

22. $\begin{cases} x^2 + y = x - y^2 + 42, \\ xy = 20. \end{cases}$

23. $\begin{cases} x + y + 2\sqrt{x + y} = 24, \\ x - y + 3\sqrt{x - y} = 10. \end{cases}$

24. $\begin{cases} x^2 + y^2 + 6\sqrt{x^2 + y^2} = 55, \\ x^2 - y^2 = 7. \end{cases}$

25. $\begin{cases} x^2 - 6xy + 9y^2 + 2x - 6y - 8 = 0, \\ x^2 + 4xy + 4y^2 - 4x - 8y - 21 = 0. \end{cases}$

SUGGESTION. — The equations may be written in the quadratic form.

Thus, $\begin{cases} (x - 3y)^2 + 2(x - 3y) - 8 = 0, \\ (x + 2y)^2 - 4(x + 2y) - 21 = 0. \end{cases}$

26. Solve $\left\{ \begin{array}{l} x^2 - xy = a^2 + b^2 \\ xy - y^2 = 2ab \end{array} \right\}$ for x and y.

27. Solve $\left\{ \begin{array}{l} x - 2y = 2(a + b) \\ xy + 2y^2 = 2b(b - a) \end{array} \right\}$ for x and y.

28. Solve $\left\{ \begin{array}{l} s = \frac{1}{2}at^2 \\ v = at \end{array} \right\}$ for a and t.

29. Solve $\left\{ \begin{array}{l} s = 6t + \frac{1}{2}at^2 \\ v = at \end{array} \right\}$ for v and t.

Problems

253. 1. The sum of two numbers is 16 and their product is 48. What are the numbers?

2. The difference between two numbers is 4 and their product is 77. Find the numbers.

3. The product of two numbers is 108 and their quotient is $1\frac{1}{3}$. Find the numbers.

4. The sum of two numbers is 8 and the sum of their squares is 40. Find the numbers.

5. The difference between two numbers is 2 and the difference between their cubes is 26. Find the numbers.

6. The sum of two numbers is 82 and the sum of their square roots is 10. What are the numbers?

7. The perimeter of a rectangle is 20 inches and its area is 24 square inches. Find its dimensions.

8. The product of two numbers is s^2 and the difference between them is 8 times the smaller number. What are the numbers?

9. The perimeter of a floor is 44 feet and its area is 120 square feet. Find its length and its width.

10. An electric sign is 10 feet longer than it is wide and its area is 6375 square feet. Find its dimensions.

11. The sum of the sides of two squares is 12. If the difference between their areas is 3, what is the side of each?

12. The area of a rectangular field is 3 acres and its length is 4 rods more than its width. Find its dimensions.

13. An Indian blanket has an area of 35 square feet. If its width were 1 foot less and its length 1 foot more, the former dimension would be $\frac{1}{2}$ of the latter. Find its dimensions.

14. The product of two numbers is 18 less than 10 times the larger number and 8 less than 10 times the smaller number. Find the numbers.

15. If a two-digit number is multiplied by its units' digit, the result is 24. If the sum of the digits is added to the number, the result is 15. What is the number?

16. The perimeter of a right triangle is 12 feet and its hypotenuse is 1 foot longer than its base. Find its base.

17. If a two-digit number is multiplied by the sum of its digits, the result is 198. If it is divided by the sum of its digits, the result is $5\frac{1}{2}$. Find the number.

18. The denominator of a certain fraction exceeds its numerator by 1, and if the fraction is multiplied by the sum of its terms, the result is $3\frac{1}{3}$. Find the fraction.

19. The base of a triangle was 7 inches longer than its altitude and its area was $\frac{1}{2}$ of a square foot. Find the dimensions of the triangle.

20. The size of an oriental prayer carpet was 23 square feet. If the width was 10 inches more than $\frac{1}{2}$ the length, what were the dimensions of the carpet?

21. The difference between two numbers is $2a$ and their product is b. Find the numbers.

22. A certain door mat has an area of 882 square inches. If its length had been 6 inches less and its width $5\frac{1}{2}$ inches more, the mat would have been square. Find its dimensions.

23. I paid 75 ¢ for ribbon. If it had cost 10 ¢ less per yard, I should have received 2 yards more for the same money. How many yards did I buy, and what was the price per yard?

24. A man expended $ 6.00 for canvas. Had it cost 4 cents less per yard, he would have received 5 yards more. How many yards did he buy, and at what price per yard?

25. The central court of the New York State Capitol has an area of 12,604 square feet. What are the dimensions of the court, if the width is 2 feet more than twice the difference between the length and the width?

26. The radius of one circle is $\frac{2}{3}$ that of another circle. If the sum of the areas of the circles is 117π square feet, how long is the radius of each circle?

27. A grocer sold carrots for \$4.40. If the number of bunches had been 4 less and the price per bunch 1 ¢ more, he would still have received \$4.40. Find the price per bunch.

28. One machine sticks 720,000 pins into the papers per day. If the machine ran 2 hours longer daily and stuck into the papers 18,000 pins less hourly, the result would be the same. How long does the machine run per day?

29. A merchant bought a piece of cloth for \$147. He cut 12 yards that were damaged from the piece and then sold the remainder for \$120.25 at a gain of 25 ¢ per yard. How many yards did he buy? What was the cost per yard?

30. A ship was loaded with 2000 tons of coal. If 50 tons more had been put on per hour, it would have taken 1 hour 20 minutes less time to load the whole amount. How long did it take to load the coal?

31. A man packed 2000 pounds of cherries in boxes. If each box had contained 6 pounds more, he would have used 75 boxes less. How many boxes did he use and how many pounds of cherries did each contain?

32. A farmer received 20 ¢ less per bushel for oats than for rye, and sold 3 bushels more of oats than of rye. The receipts from the oats were \$4.50 and from the rye \$4.20. Find the number of bushels of each sold and the price per bushel.

33. Three men earned \$87.36. If A had worked 3 days less he would have earned the same as B; if $2\frac{1}{2}$ times as long he would have earned the same as C. C earned \$16.64 more than A and B together. Find the daily wages of each.

34. A boy has a large blotter, 4 inches longer than it is wide, and 480 square inches in area. He wishes to cut away enough to leave a square 256 square inches in area. How many inches must he cut from the length and from the width?

35. The total area of a rug whose length is 3 feet more than its width is 108 square feet. The area of the rug exclusive of the border is 54 square feet. Find the width of the border.

36. After a mowing machine had made the circuit of a 7-acre rectangular hay field 11 times, cutting a swath 6 feet wide each time, 4 acres of grass were still standing. Find the dimensions of the field in rods.

37. The amount of a sum of money for one year is $ 3990. If the rate were 1 % less and the principal were $ 200 more, the amount would be $ 4160. Find the principal and the rate.

38. My annual income from an investment is $ 60. If the principal were $ 500 less and the rate of interest 1 % more, my income would be the same. Find the principal and the rate.

39. A sum of money on interest for one year at a certain per cent amounted to $ 11,130. If the rate had been 1 % less and the principal $ 100 more, the amount would have been the same. Find the principal and the rate.

40. The fore wheel of a carriage makes 12 revolutions more than the hind wheel in going 240 yards. If the circumference of each wheel were 1 yard greater, the fore wheel would make 8 revolutions more than the hind wheel in going 240 yards. What is the circumference of each wheel?

41. The town A is on a lake and 12 miles from B, which is 4 miles from the opposite shore. A man rows across the lake and walks to B in 3 hours. Returning, he walks at the same rate, but rows 2 miles an hour less than before. It takes him 5 hours to return. Find his rates of rowing and walking.

42. A, B, and C started at the same time to ride a certain distance. A and C rode the whole distance at uniform rates, A 2 miles an hour faster than C. B rode with C for 20 miles, and then by increasing his speed 2 miles an hour, reached his destination 40 minutes before C and 20 minutes after A. Find the distance and the rate at which each traveled.

GRAPHIC SOLUTIONS

QUADRATIC FUNCTIONS

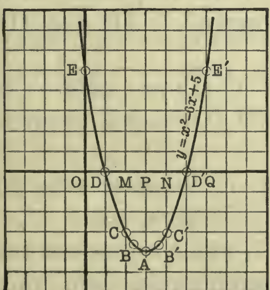
254. Graphic solutions of quadratic equations in *x*.

Let it be required to solve graphically, $x^2 - 6x + 5 = 0$.

To do this, we must construct the graph of $f(x) = x^2 - 6x + 5$, that is, of $y = x^2 - 6x + 5$. The graph will represent *all* the corresponding real values of x and of $x^2 - 6x + 5$, and among them will be the values of x that make $x^2 - 6x + 5$ equal to zero, that is, the *roots* of the equation $x^2 - 6x + 5 = 0$.

When the coefficient of x^2 is $+1$, as in this instance, it is convenient to take for the first value of x a number equal to half the coefficient of x with its sign changed. Next, values of x differing from this value *by equal amounts* may be taken.

Thus, first substituting $x = 3$, it is found that $y = -4$, locating the point $A = (3, -4)$. Next give values to x differing from 3 by equal amounts, as $2\frac{1}{2}$ and $3\frac{1}{2}$, 2 and 4, 1 and 5, 0 and 6. It will be found that y has the same value for $x = 3\frac{1}{2}$ as for $x = 2\frac{1}{2}$, for $x = 4$ as for $x = 2$, etc. The table below gives a record of the points and their coördinates.



x	y	POINTS
3	-4	A
$2\frac{1}{2}$, $3\frac{1}{2}$	$-3\frac{3}{4}$	B, B'
2, 4	-3	C, C'
1, 5	0	D, D'
0, 6	5	E, E'

Plotting the points A; B, B'; C, C'; etc., whose coördinates are given in the preceding table, and drawing a smooth curve through them, we obtain the graph of $y = x^2 - 6x + 5$ as shown in the figure.

Observe from the preceding graph and table that:

When $x = 3$, $x^2 - 6x + 5 = -4$, which is represented by the *negative* ordinate PA.

When $x = 2$ and also when $x = 4$, $x^2 - 6x + 5 = -3$, which is represented by the equal *negative* ordinates MC and NC'.

When $x = 0$ and also when $x = 6$, $x^2 - 6x + 5 = 5$, represented by the equal *positive* ordinates OE and QE'.

The ordinates change sign as the curve crosses the x-axis.

At D and at D', where the ordinates are equal to 0, the value of $x^2 - 6x + 5$ is 0, and the abscissas are $x = 1$ and $x = 5$.

Hence, the roots of the given equation are 1 and 5.

NOTE. — Half the coefficient of x with its sign changed, the number first substituted for x, is half the sum of the roots, or their *mean value*, when the coefficient of x^2 is $+1$. This will be shown in § 266.

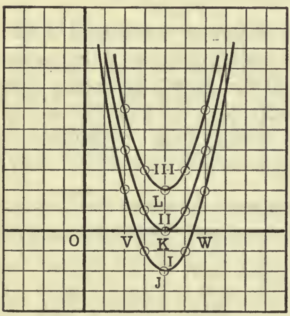
The curve obtained by plotting the graph of any quadratic function of the form $ax^2 + bx + c$ is a **parabola**.

255. Let it be required to solve each of the equations

$$x^2 - 8x + 14 = 0, \qquad (1)$$

$$x^2 - 8x + 16 = 0, \qquad (2)$$

$$x^2 - 8x + 18 = 0. \qquad (3)$$



The graphs corresponding to equations (1), (2), and (3), found as in § 254, are marked I, II, and III, respectively.

The roots of (1) are seen to be $OV = 2.6$ and $OW = 5.4$, approximately.

Since graph II has only one point, K, in common with the x-axis, equation (2) appears to have only one root, $OK = 4$.

But it will be observed that if graph I, which represents two unequal real roots, OV and OW, were moved upward two units, it would coincide with graph II. During this process the unequal roots of (1), OV and OW, would approach the value OK, which represents the roots of (2).

Consequently, the roots of (2) are regarded as *two* in number. They are *real* and *equal*, or *coincident*.

The movement of the graph of (1) *upward* the distance JK, or 2 units, corresponds to completing the square in (1) by *adding* 2 to each member. Since the roots of the resulting equation, $x^2 - 8x + 16 = 2$, differ from those of (2) or from the mean value $OK \doteq 4$, by $\pm\sqrt{2}$, or $\pm\sqrt{JK}$, it is evident that the roots of (1) are represented graphically by

$$OK + \sqrt{JK} = 4 + \sqrt{2} = 5.414^+,$$

and

$$OK - \sqrt{JK} = 4 - \sqrt{2} = 2.586^-.$$

Since graph III has no point on the x-axis, there are no real values of x for which $x^2 - 8x + 18$ is equal to zero; that is, (3) has no real roots. Consequently, the roots are *imaginary*.

If graph III were moved *downward* 2 units, it would coincide with graph II. If the square in (3) were completed by *subtracting* 2 from each member, the roots of the resulting equation, $x^2 - 8x + 16 = -2$, would differ from the mean value by $\pm\sqrt{-2}$, or $\pm\sqrt{LK}$.

Hence, it is evident that the roots of (3) are represented graphically by

$$OK + \sqrt{LK} = 4 + \sqrt{-2},$$

and

$$OK - \sqrt{LK} = 4 - \sqrt{-2}.$$

The points J, K, and L, whose ordinates are the least algebraically that any points in the respective graphs can have, are called **minimum points**.

256. When the coefficient of x^2 is $+1$, it is evident from the preceding discussion that:

PRINCIPLES. — 1. *The roots of a quadratic in x are equal to the abscissa of the minimum point, plus or minus the square root of the ordinate with its sign changed.*

2. *If the minimum point lies on the x-axis, the roots are real and equal.*

3. *If the minimum point lies below the x-axis, the roots are real and unequal.*

4. *If the minimum point lies above the x-axis, the roots are imaginary.*

EXERCISES

257. Solve graphically, giving real roots to the nearest tenth.

1. $x^2 + x - 2 = 0$.

2. $x^2 - x + 6 = 0$.

3. $x^2 - 3x - 4 = 0$.

4. $x^2 - 2x - 15 = 0$.

5. $x^2 + 5x + 14 = 0$.

6. $x^2 + 3x - 10 = 0$.

7. $x^2 - 7x + 18 = 0$.

8. $x^2 + 4x + 45 = 0$.

9. $x^2 + 6x - 27 = 0$.

10. $x^2 - 14x - 51 = 0$.

11. $2x^2 - x - 6 = 0$.

SUGGESTION. — Reduce the equation to the form $x^2 + px + q = 0$, in which the coefficient of x^2 is $+1$, and proceed as in the exercises above.

12. $2x^2 - x - 15 = 0$.

13. $3x^2 + 5x - 28 = 0$.

14. $6x^2 - 7x = 20$.

15. $8x^2 + 14x = 15$.

258. Graphs of quadratic equations in *x* and *y*.

EXERCISES

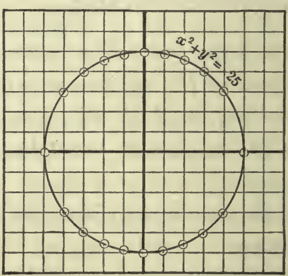
1. Construct the graph of the equation $x^2 + y^2 = 25$.

SOLUTION. — Solve for y, $y = \pm\sqrt{25 - x^2}$.

Since any value numerically greater than 5 substituted for x will make the value of y imaginary, we substitute only values of x between and including -5 and $+5$. The corresponding values of x and y, or $\pm\sqrt{25 - x^2}$, are recorded in the table below.

It will be observed that each value substituted for x, except ± 5, gives two values of y, and that values of x numerically equal give the same values of y; thus, when $x=2$, $y=\pm 4.6$, and also when $x=-2$, $y=\pm 4.6$.

x	y
0	± 5
± 1	± 4.9
± 2	± 4.6
± 3	± 4
± 4	± 3
± 5	0



The values given in the table serve to locate twenty points of the

graph of $x^2 + y^2 = 25$. Plotting these points and drawing a smooth curve through them, we see that the graph is apparently a *circle*. It may be proved by geometry that this graph is a circle whose radius is 5.

The graph of any equation of the form $x^2 + y^2 = r^2$ is a **circle** whose radius is r and whose center is at the origin.

2. Construct the graph of the equation $x^2 + y^2 = 49$.

3. Construct the graph of the equation $(x-2)^2 + (y-3)^2 = 9$.

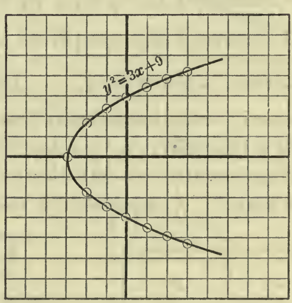
Suggestion. — Solving for y, we have $y = 3 \pm \sqrt{9 - (x-2)^2}$.
Since any value less than -1 or greater than $+5$ substituted for x makes the value of y imaginary, the graph lies between $x = -1$ and $+5$.

The graph of any equation of the form $(x-a)^2 + (y-b)^2 = r^2$ is a **circle** whose radius is r and center is at the point (a, b).

4. Construct the graph of the equation $y^2 = 3x + 9$.

Solution. — Solve for y, $y = \pm \sqrt{3x + 9}$.
It will be observed that any value smaller than -3 substituted for x will make y imaginary; consequently, no point of the graph lies to the left of $x = -3$. Beginning with $x = -3$, we substitute values for x and determine the corresponding values of y, as recorded in the table.

x	y
-3	0
-2	± 1.7
-1	$\pm 2\ 4$
0	± 3
1	± 3.5
2	± 3.9
3	± 4.2



Plotting these points and drawing a smooth curve through them, we find that the graph obtained is apparently a *parabola*.

The graph of any equation of the form $y^2 = ax + c$ is a **parabola**.

5. Construct the graph of $y^2 = 5x + 8$.

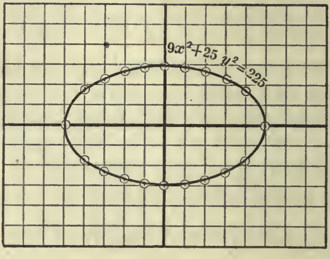
6. Construct the graph of the equation $9x^2 + 25y^2 = 225$.

SOLUTION. — Solve for y, $y = \pm \frac{3}{5}\sqrt{25 - x^2}$.

Since any value numerically greater than 5 substituted for x will make the value of y imaginary, no point of the graph lies farther to the right or to the left of the origin than 5 units; consequently, we substitute for x only values between and including -5 and $+5$.

Corresponding values of x and y are given in the table.

x	y
0	± 3
± 1	± 2.9
± 2	± 2.7
± 3	± 2.4
± 4	± 1.8
± 5	0



Plotting these twenty points and drawing a smooth curve through them, we have the graph of $9x^2 + 25y^2 = 225$, which is called an *ellipse.*

The graph of any equation of the form $b^2x^2 + a^2y^2 = a^2b^2$ is an **ellipse.**

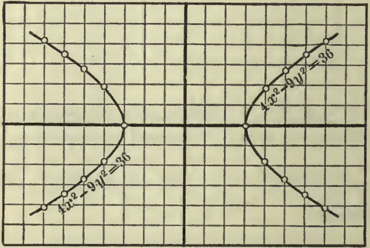
7. Construct the graph of the equation $9x^2 + 16y^2 = 144$.

8. Construct the graph of the equation $4x^2 - 9y^2 = 36$.

SOLUTION. — Solve for y, $\quad y = \pm \frac{2}{3}\sqrt{x^2 - 9}$.

Since any value numerically less than 3 substituted for x will make the value of y imaginary, no point of the graph lies between $x = +3$ and $x = -3$; consequently, we substitute for x only ± 3 and values numerically greater than 3. Corresponding values of x and y are given in the table.

x	y
± 3	0
± 4	± 1.8
± 5	± 2.7
± 6	± 3.5
± 7	± 4.2



Plotting these eighteen points, we find that half of them are on one

side of the y-axis and half on the other side, and since there are no points of the curve between $x = +3$ and $x = -3$, the graph has two separate *branches*, that is, it is *discontinuous*.

Drawing a smooth curve through each group of points, we see that the two branches thus constructed constitute the graph of the equation $4x^2 - 9y^2 = 36$, which is an *hyperbola*.

The graph of any equation of the form $b^2x^2 - a^2y^2 = a^2b^2$ is an **hyperbola**. An hyperbola has two **branches** and is called a **discontinuous** curve.

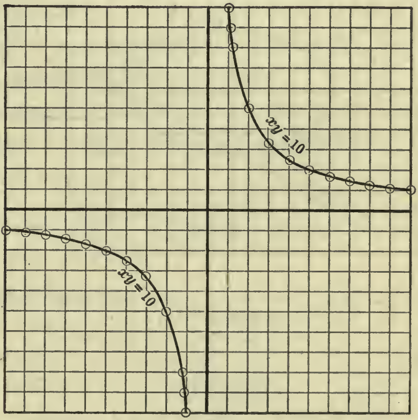
9. Construct the graph of the equation $9x^2 - 16y^2 = 144$.

10. Construct the graph of the equation $xy = 10$.

SOLUTION

Substituting values for x and solving for y, we find the corresponding values of x and y as given in the table.

x	y	x	y
1	10	-1	-10
2	5	-2	-5
3	$3\frac{1}{3}$	-3	$-3\frac{1}{3}$
4	$2\frac{1}{2}$	-4	$-2\frac{1}{2}$
5	2	-5	-2
6	$1\frac{2}{3}$	-6	$-1\frac{2}{3}$
7	$1\frac{3}{7}$	-7	$-1\frac{3}{7}$
8	$1\frac{1}{4}$	-8	$-1\frac{1}{4}$
9	$1\frac{1}{9}$	-9	$-1\frac{1}{9}$
10	1	-10	-1



Plotting these points and drawing a smooth curve through each group of points, we see that the two branches of the curve found constitute the graph of the equation $xy = 10$, which is an *hyperbola*.

The graph of any equation of the form $xy = c$ is an **hyperbola**.

11. Construct the graph of the equation $xy = 12$.

12. Construct the graph of the equation $xy = -12$.

259. Graphic solutions of simultaneous equations involving quadratics.

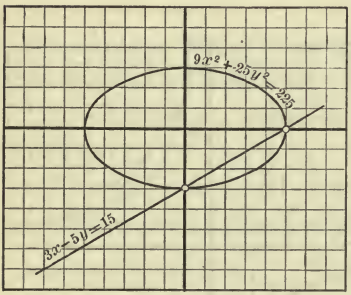
The graphic method of solving simultaneous equations that involve quadratics is precisely the same as for simultaneous linear equations (§§ 149–153), namely:

Construct the graph of each equation, both being referred to the same axes, and determine the coördinates of the points where the graphs intersect. If they do not intersect, interpret this fact.

EXERCISES

260. 1. Solve graphically $\begin{cases} 9x^2 + 25y^2 = 225, \\ 3x - 5y = 15. \end{cases}$

SOLUTION



Constructing the graphs of these equations, we find the first to be an ellipse and the second a straight line.

The straight line intersects the ellipse in two points, (5, 0) and (0, – 3).

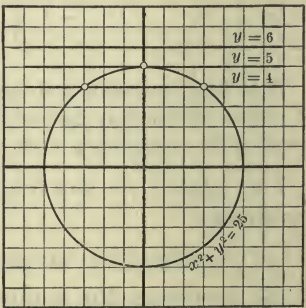
Hence, there are two solutions,

$x = 5,\ y = 0$; and $x = 0,\ y = -3$.

TEST. — The student may test the roots found by performing the numerical solution.

2. Solve graphically $\begin{cases} x^2 + y^2 = 25, \\ y = 4. \end{cases}$

SOLUTION



The graphs (a circle and a straight line) are found to intersect at the points,

$x = 3,\ y = 4$; $x = -3,\ y = 4$.

Since the graphs have *only* these two points in common, their coördinates are the only values of x and y that satisfy both equations, and are the roots sought.

The pairs of values found are *real*, and different, or *unequal*.

3. Solve graphically $\begin{cases} x^2 + y^2 = 25, \\ y = 5. \end{cases}$

Solution. — Imagine the straight line $y = 4$ in the figure for exercise 2 to move upward until it coincides with the line $y = 5$. The real unequal roots represented by the coördinates of the points of intersection approach equality, and when the line becomes the tangent line $y = 5$, they coincide.

Hence, the given system of equations has *two real equal roots*, $x = 0$, $y = 5$, and $x = 0$, $y = 5$.

4. Find the nature of the roots of $\begin{cases} x^2 + y^2 = 25, \\ y = 6. \end{cases}$

Solution. — Imagine the straight line $y = 4$, in the figure for exercise 2 to move upward until it coincides with the line $y = 6$. The graphs will cease to have any points in common, showing that the given equations have *no common real values* of x and y.

It is shown by the numerical solution of the equations that there are two roots and that both are *imaginary*.

A system of two independent simultaneous equations in x and y, one simple and the other quadratic, has two roots.

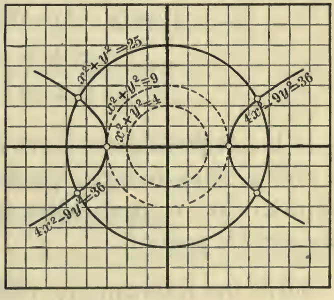
The roots are real and unequal if the graphs intersect, real and equal if the graphs are tangent to each other, and imaginary if the graphs have no points in common.

5. Solve graphically $\begin{cases} 4x^2 - 9y^2 = 36, \\ x^2 + y^2 = 25. \end{cases}$

Solution. — The graphs (the first an hyperbola and the second a circle) show that both of the given equations are satisfied by *four* different pairs of real values of x and y:

$$\begin{cases} x = 4.5\ ;\ 4.5\ ;\ -4.5\ ;\ -4.5\ ; \\ y = 2.2\ ;\ -2.2\ ;\ -2.2;\ 2.2. \end{cases}$$

Note. — The roots are estimated to the nearest tenth; their accuracy may be tested by performing the numerical solution.



6. What would be the nature of the roots in exercise 5, if the second equation were $x^2 + y^2 = 9$? $x^2 + y^2 = 4$?

A system of two independent simultaneous quadratic equations in x and y has four roots.

An intersection of the graphs represents a real root, and a point of tangency, a pair of equal real roots. If there are less than four real roots, the other roots are imaginary.

Find by graphic methods, to the nearest tenth, the real roots of the following, and the number of imaginary roots, if there are any. Discuss the graphs and the roots:

7. $\begin{cases} x^2 + y^2 = 36, \\ 2x - 3y = 6. \end{cases}$

8. $\begin{cases} 2x + 5y = 10, \\ 5x^2 + 2y^2 = 125. \end{cases}$

9. $\begin{cases} x - 3y = 2, \\ 9x^2 - 16y^2 = 144. \end{cases}$

10. $\begin{cases} xy = -1, \\ x - y = 2. \end{cases}$

11. $\begin{cases} x^2 - 4y^2 = 1, \\ x + y = 1. \end{cases}$

12. $\begin{cases} xy = -2, \\ x^2 + 4y^2 = 17. \end{cases}$

13. $\begin{cases} x^2 + y^2 = 25, \\ (x-3)^2 + (y-4)^2 = 16. \end{cases}$

14. $\begin{cases} y^2 = 4x + 8, \\ x^2 + y^2 = 9. \end{cases}$

15. $\begin{cases} x^2 + y^2 = 16, \\ 9x^2 + 16y^2 = 144. \end{cases}$

16. $\begin{cases} x^2 + y^2 = 49, \\ x^2 - y^2 = 64. \end{cases}$

17. $\begin{cases} x^2 - y^2 = 49, \\ x^2 + y^2 = 64. \end{cases}$

18. $\begin{cases} y = x^2 - 3x + 2, \\ x = 2y^2 - 3. \end{cases}$

19. $\begin{cases} x = y^2 - 4, \\ y = (x+1)(x+4). \end{cases}$

20. $\begin{cases} x = y^2 - 5y + 4, \\ y = x^2 - 4y + 3. \end{cases}$

21. $\begin{cases} y^2 + x^2 + y - 2x + 1 = 0, \\ y^2 + x^2 + 3y - 4x + 3 = 0. \end{cases}$

It is not possible to solve *any* two simultaneous equations in x and y, that involve quadratics, *by quadratic methods*, but approximate values of the real roots may always be found *by the graphic method.*

Solve the following by both methods, if you can:

22. $\begin{cases} x^2 + y^2 = 26, \\ x^2y + y = 26. \end{cases}$

23. $\begin{cases} x^2 + y = 7, \\ y^2 + x = 11. \end{cases}$

PROPERTIES OF QUADRATIC EQUATIONS

261. Nature of the roots.

In the following discussion the student should keep in mind the distinctions between **rational** and **irrational, real** and **imaginary.**

For example, 2 and $\sqrt{4}$ are *rational* and also *real;* $\sqrt{2}$ and $\sqrt{5}$ are *irrational*, but *real;* $\sqrt{-2}$ and $\sqrt{-5}$ are *irrational* and also *imaginary.*

262. Every quadratic equation may be reduced to the form

$$ax^2 + bx + c = 0,$$

in which a is positive and b and c are positive or negative.

Denote the roots by r_1 and r_2. Then, § 226,

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

An examination of the above values of r_1 and r_2 will show that the nature of the roots, as real or imaginary, rational or irrational, may be determined by observing whether $\sqrt{b^2 - 4ac}$ is real or imaginary, rational or irrational. Hence,

PRINCIPLES. — In any quadratic equation, $ax^2 + bx + c = 0$, when a, b, and c represent real and rational numbers:

1. *If $b^2 - 4ac$ is **positive**, the roots are **real** and **unequal**.*

2. *If $b^2 - 4ac$ equals **zero**, the roots are **real** and **equal**.*

3. *If $b^2 - 4ac$ is **negative**, the roots are **imaginary**.*

4. *If $b^2 - 4ac$ is a **perfect square** or equals **zero**, the roots are **rational**; otherwise, they are irrational.*

263. The expression $b^2 - 4ac$ is called the **discriminant** of the quadratic equation $ax^2 + bx + c = 0$.

264. If a is positive and b and c are positive or negative, the **signs of the roots** of $ax^2 + bx + c = 0$, that is, the signs of

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a},$$

may be determined from the signs of b and c.

Thus, if c is positive, $-b$ is numerically greater than $\pm\sqrt{b^2 - 4ac}$, whence both roots have the sign of $-b$; if c is negative, $-b$ is numerically less than $\pm\sqrt{b^2 - 4ac}$, whence r_1 is positive and r_2 is negative. The root having the sign opposite to that of b is the greater numerically. Hence,

Principle. — *If c is positive, both roots have the sign opposite to that of b; if c is negative, the roots have opposite signs, and the numerically greater root has the sign opposite to that of b.*

Note. — If $b = 0$, the roots have opposite signs. (See also § 219.)

EXERCISES

265. 1. What is the nature of the roots of $x^2 - 7x - 8 = 0$?

Solution. — Since $b^2 - 4ac = 49 + 32 = 81 = 9^2$, a positive number and a perfect square, by § 262, Prin. 1, the roots are real and unequal; and by Prin. 4, rational.

Since c is negative, by § 264, Prin., the roots have opposite signs and, b being negative, the positive root is the greater numerically.

2. What is the nature of the roots of $3x^2 + 5x + 3 = 0$?

Solution. — Since $b^2 - 4ac = 25 - 36 = -11$, a negative number, by § 262, Prin. 3, both roots are imaginary.

Find, without solving, the nature of the roots of:

3. $x^2 - 5x - 75 = 0$.
4. $x^2 + 5x + 6 = 0$.
5. $x^2 + 7x - 30 = 0$.
6. $x^2 - 3x + 5 = 0$.
7. $x^2 + 3x - 5 = 0$.
8. $4x^2 - 4x + 1 = 0$.
9. $4x^2 + 6x - 4 = 0$.
10. $x^2 + x + 2 = 0$.
11. $4x^2 + 16x + 7 = 0$.
12. $9x^2 + 12x + 4 = 0$.

13. For what values of m will the equation

$$2x^2 + 3mx + 2 = 0$$

have equal roots? imaginary roots?

SOLUTION

The roots will be equal, if the discriminant equals zero (§ 262, Prin. 2); that is, if $(3m)^2 - 4 \cdot 2 \cdot 2 = 0$,
or, solving, if $m = \frac{4}{3}$ or $-\frac{4}{3}$.

The roots will be imaginary, if the discriminant is negative (§ 262, Prin. 3); that is, if $(3m)^2 - 4 \cdot 2 \cdot 2$ is negative,
which will be true when m is numerically less than $\frac{4}{3}$.

14. For what values of m will $9x^2 - 5mx + 25 = 0$ have equal roots? real roots? imaginary roots?

15. For what values of a will the roots of the equation

$$4x^2 - 2(a - 3)x + 1 = 0$$

be real and equal? real and unequal? imaginary?

16. Find the values of m for which the roots of the equation

$$4x^2 + mx + x + 1 = 0$$

are equal. What are the corresponding values of x?

17. For what values of n are the roots of the equation
$3x^2 + 1 = n(4x - 2x^2 - 1)$ real and equal?

18. For what value of a are the roots of the equation

$$ax^2 - (a - 1)x + 1 = 0$$

numerically equal but opposite in sign? Find the roots for this value of a.

19. For what values of d has $x^2 + (2 - d)x = 3d^2 - 27$ a zero root? Find both roots for each of these values of d.

20. For what values of m will the roots of the equation

$(m + \frac{5}{2})x^2 - 2(m + 1)x + 2 = 0$ be equal?

21. Solve the simultaneous equations for x and y

$$\begin{cases} 3x^2 - 4y^2 = 8, \\ 5(x - k) - 4y = 0. \end{cases}$$

For what values of k are the roots real? imaginary? equal?

266. Relation of roots and coefficients.

Any quadratic equation, as $ax^2 + bx + c = 0$, may be reduced, by dividing both members by the coefficient of x^2, to the form $x^2 + px + q = 0$, whose roots by actual solution are found to be

$$r_1 = \frac{-p + \sqrt{p^2 - 4q}}{2} \text{ and } r_2 = \frac{-p - \sqrt{p^2 - 4q}}{2}.$$

Add the roots, $r_1 + r_2 = \dfrac{-2p}{2} = -p.$

Multiply the roots, $r_1 r_2 = \dfrac{p^2 - (p^2 - 4q)}{4} = q.$

Hence, we have the following:

PRINCIPLE. — *The sum of the roots of a quadratic equation having the form $x^2 + px + q = 0$ is equal to the coefficient of x with its sign changed, and their product is equal to the absolute term.*

267. Formation of quadratic equations.

Substituting $-(r_1 + r_2)$ for p, and $r_1 r_2$ for q (§ 266) in the equation $x^2 + px + q = 0$, we have

$$x^2 - (r_1 + r_2)x + r_1 r_2 = 0.$$

Expand, $\quad x^2 - r_1 x - r_2 x + r_1 r_2 = 0.$

Factor, $\quad (x - r_1)(x - r_2) = 0.$

Hence, to form a quadratic equation whose roots are given:

√*Subtract each root from x and place the product of the remainders equal to zero.*

EXERCISES

268. 1. Form an equation whose roots are -5 and 2.

SOLUTION. $(x + 5)(x - 2) = 0$, or $x^2 + 3x - 10 = 0$.

Or, since the sum of the roots with their signs changed is $+5 - 2$, or 3, and the product of the roots is -10 (§ 266), the equation is $x^2 + 3x - 10 = 0$.

Form the equation whose roots are:

2. $6, 4$.
3. $5, -3$.
4. $3, -\frac{1}{3}$.
5. $\frac{2}{3}, \frac{5}{3}$.
6. $-2, -\frac{1}{2}$.
7. $-\frac{1}{2}, -\frac{3}{2}$.
8. $a, -3a$.
9. $a+2, a-2$.
10. $b+1, b-1$.
11. $a+b, a-b$.
12. $\sqrt{a}-\sqrt{b}, \sqrt{b}$.
13. $\frac{1}{2}(a \pm \sqrt{b})$.
14. $3+\sqrt{2}, 3-\sqrt{2}$.
15. $2-\sqrt{5}, 2+\sqrt{5}$.
16. $2 \pm \sqrt{3}$.
17. $-\frac{1}{2}(3 \pm \sqrt{6})$.
18. $\frac{1}{2}(-1 \pm \sqrt{2})$.
19. $a(2 \pm 2\sqrt{5})$.

20. What is the sum of the roots of $2m^2x^2-(5m-1)x=6$? For what values of m is the sum equal to 2?

21. When one of the roots of $ax^2+bx+c=0$ is twice the other, what is the relation of b^2 to a and c?

Solution

Writing $ax^2+bx+c=0$ in the form

$$x^2+\frac{b}{a}x+\frac{c}{a}=0, \tag{1}$$

and representing the roots by r and $2r$, we have

$$r+2r=3r=-\frac{b}{a}, \tag{2}$$

and
$$r \cdot 2r = 2r^2 = \frac{c}{a}. \tag{3}$$

On substituting the value of r obtained from (2) in (3) and reducing,

$$b^2=\frac{9}{2}ac.$$

22. Obtain an equation expressing the condition that one root of $4x^2-3ax+b=3$ is twice the other.

23. Find the condition that one root of $ax^2+bx+c=0$ shall be greater than the other by 3.

24. When one root of the general quadratic equation $ax^2+bx+c=0$ is the reciprocal of the other, what is the relation between a and c?

25. If the roots of $ax^2+bx+c=0$ are r_1 and r_2, write an equation whose roots are $-r_1$ and $-r_2$.

26. Obtain the sum of the squares of the roots of $2x^2-12x+3=0$, without solving the equation.

SOLUTION

Sum of roots $= r_1 + r_2 = 6.$ (1)

Product of roots $= r_1r_2 = \frac{3}{2}.$ (2)

Square (1), $r_1^2 + r_2^2 + 2r_1r_2 = 36.$ (3)

(2) × 2, $2r_1r_2 = 3.$ (4)

(3) − (4), $r_1^2 + r_2^2 = 33.$

Find, without solving the equation:

27. The sum of the squares of the roots of $x^2-5x-6=0.$

28. The sum of the cubes of the roots of $2x^2-3x+1=0.$

29. The difference between the roots of $12x^2+x-1=0.$

30. The square root of the sum of the squares of the roots of $x^2-7x+12=0.$

31. The sum of the reciprocals of the roots of $ax^2+bx+c=0.$

SUGGESTION. $\frac{1}{r_1}+\frac{1}{r_2}=\frac{r_1+r_2}{r_1r_2}.$

32. The difference between the reciprocals of the roots of $8x^2-10x+3=0.$

269. The number of roots of a quadratic equation.

It has been seen (§ 266) that any quadratic equation may be reduced to the form $x^2+px+q=0$, which has *two* roots, as r_1 and r_2. To show that the equation cannot have more than two roots, write it in the form given in § 267, namely,

$$(x-r_1)(x-r_2)=0. \quad (1)$$

If the equation has a third root, suppose it is r_3.

Substituting r_3 for x in (1), we have

$$(r_3-r_1)(r_3-r_2)=0,$$

which is impossible, if r_3 differs from both r_1 and r_2. Hence,

PRINCIPLE. — *A quadratic equation has two and only two roots.*

270. Factoring by completing the square.

The method of factoring is useful in solving quadratic equations when the factors are rational and readily seen. In more difficult cases we complete the square. This more powerful method is useful also in factoring quadratic expressions the factors of which are irrational or otherwise difficult to obtain.

EXERCISES

271. 1. Factor $2x^2 + 5x - 3$.

SOLUTION. — Let $2x^2 + 5x - 3 = 0$.
Divide by 2, etc., $x^2 + \frac{5}{2}x = \frac{3}{2}$.
Complete the square, $x^2 + \frac{5}{2}x + \frac{25}{16} = \frac{49}{16}$.
Solve, $x = \frac{1}{2}$ or -3.
Forming an equation having these roots, § 267, we have

$$(x - \tfrac{1}{2})(x + 3) = 0.$$

Multiplying by 2 because we divided by 2, we have

$$(2x - 1)(x + 3) = 2x^2 + 5x - 3 = 0.$$

Hence, the factors of $2x^2 + 5x - 3$ are $2x - 1$ and $x + 3$.

Factor:

2. $5x^2 + 3x - 2$.
3. $4x^2 - 4x - 3$.
4. $8x^2 - 14x + 3$.
5. $7x^2 + 13x - 2$.
6. $15x^2 - 5.5x - 1$.
7. $24x^2 - 10x - 25$.

8. Factor $x^2 + 2x - 4$.

SOLUTION. — Let $x^2 + 2x - 4 = 0$.
Complete the square, $x^2 + 2x + 1 = 5$.
Solve, $x = -1 + \sqrt{5}$ or $-1 - \sqrt{5}$.
Hence, § 267, $(x + 1 - \sqrt{5})(x + 1 + \sqrt{5}) = x^2 + 2x - 4 = 0$.
That is, the factors of $x^2 + 2x - 4$ are $x + 1 - \sqrt{5}$ and $x + 1 + \sqrt{5}$.

9. $x^2 + 4x - 6$.
10. $y^2 - 6y + 3$.
11. $z^2 - 5z - 1$.
12. $x^2 + x + 1$.
13. $t^2 + 3t + 7$.
14. $a^2 + 3a - 5$.

15. Factor $2 - 3x - 2x^2$.

SUGGESTION. — Since $2 - 3x - 2x^2 = -2(x^2 + \frac{3}{2}x - 1)$, factor $x^2 + \frac{3}{2}x - 1$, in which the coefficient of x^2 is $+1$, and multiply the result by -2.

Factor:

16. $2x^2 + 2x - 1$.

17. $9x^2 - 4x + 1$.

18. $24x - 16x^2 - 3$.

19. $9a^2 - 12a + 5$.

20. $16v(1 - v) - 9$.

21. $16(3 + n) + 3n^2$.

22. Factor $100x^2 + 70xy - 119y^2$.

Suggestion. — The coefficient of x^2 being a perfect square, complete the square directly; do not divide by 100.

23. $4b^2 - 48b + 143$.

24. $9r^2 - 12r + 437$.

25. $4a^2 + 12a - 135$.

26. $16p(p + 1) - 1517$.

27. $25e^2 - 2h(5e - 2h)$.

28. $3h(4k - 3h) - 7k^2$.

29. Factor $x^4 + 4x^3 + 8x^2 + 8x - 5$.

Solution. — Let $x^4 + 4x^3 + 8x^2 + 8x - 5 = 0$.

Complete the square,

$$(x^4 + 4x^3 + 4x^2) + 4(x^2 + 2x) + 4 = 9.$$

Extract the square root, $x^2 + 2x + 2 = 3 \text{ or } -3$.

$$\begin{aligned} \therefore x^4 + 4x^3 + 8x^2 + 8x - 5 &= (x^2 + 2x + 2 - 3)(x^2 + 2x + 2 + 3) \\ &= (x^2 + 2x - 1)(x^2 + 2x + 5). \end{aligned}$$

Factor the following polynomials:

30. $x^4 + 6x^3 + 11x^2 + 6x - 8$.

31. $x^6 + 2x^5 + 5x^4 + 8x^3 + 8x^2 + 8x + 3$.

32. $x^6 - 4x^5 + 6x^4 + 6x^3 - 19x^2 + 10x + 9$.

33. $4x^6 + 12x^5 + 25x^4 + 40x^3 + 40x^2 + 32x + 15$.

34. Resolve $x^4 + 1$ into factors of the second degree.

Solution.

$$\begin{aligned} x^4 + 1 &= x^4 + 2x^2 + 1 - 2x^2 \\ &= (x^2 + 1)^2 - (x\sqrt{2})^2 \\ &= (x^2 + x\sqrt{2} + 1)(x^2 - x\sqrt{2} + 1). \end{aligned}$$

Note. — Each of these quadratic factors may be resolved into two factors of the first degree by completing the square.

Resolve into quadratic factors:

35. $x^4 + 16$.

36. $a^4 + b^4$.

37. $x^4 + 2a^2x^2 + 4a^4$.

38. $v^4 - 4n^2v^2 - 2n^4$.

INTERPRETATION OF RESULTS

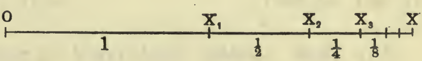
272. A number that has the same value throughout a discussion is called a **constant**.

Arithmetical numbers are constants. A literal number is constant in a discussion, if it keeps the same value throughout that discussion.

273. A number that under the conditions imposed upon it may have a series of different values is called a **variable**.

The numbers .3, .33, .333, .3333, . . . are successive values of a variable approaching in value the constant $\frac{1}{3}$.

274. When a variable takes a series of values that approach nearer and nearer a given constant without becoming equal to it, so that by taking a sufficient number of steps the difference between the variable and the constant can be made numerically less than any conceivable number however small, the constant is called the **limit** of the variable, and the variable is said to **approach its limit.**



This figure represents graphically a variable x approaching its limit $OX = 2$. The first value is $OX_1 = 1$; the second is $OX_2 = 1\frac{1}{2}$; the third is $OX_3 = 1\frac{3}{4}$; etc.

At each step the difference between the variable and its limit is diminished by half of itself. Consequently, by taking a sufficient number of steps this difference may become less than any number, however small, that may be assigned.

275. A variable that may become numerically greater than any assignable number is said to be **infinite.**

The *symbol* of an infinite number is ∞.

276. A variable that may become numerically less than any assignable number is said to be **infinitesimal.**

An infinitesimal is a variable whose limit is zero.

The character 0 is used as a symbol for an *infinitesimal number* as well as for *absolute zero,* which is the result obtained by subtracting a number from itself.

277. A number that cannot become either infinite or infinitesimal is said to be **finite.**

THE FORMS $a \times 0$, $\frac{a}{0}$, $\frac{0}{0}$, $\frac{a}{\infty}$

278. The results of algebraic processes may appear in the forms, $a \times 0$, $\frac{a}{0}$, $\frac{0}{0}$, $\frac{a}{\infty}$, etc., which are arithmetically meaningless; consequently, it becomes important to interpret the meaning of such forms.

279. Interpretation of $a \times 0$.

1. Let 0 represent *absolute zero,* defined by the identity,

$$0 = n - n. \qquad (1)$$

Multiplying $a = a$ by (1), member by member, Ax. 3, we have

$$a \times 0 = a(n - n)$$

$$= an - an$$

by def. of zero, $\qquad = 0.$ That is,

Any finite number multiplied by zero is equal to zero.

2. Let 0 represent an *infinitesimal,* as the variable whose successive values are 1, .1, .01, .001, $\cdots$.

Then, the successive values of $a \times 0$ are (§ 20)

$$a,\ .1\,a,\ .01\,a,\ .001\,a,\ \cdots. \quad \text{Hence,}$$

$a \times 0$ is a variable whose *limit* is absolute zero. That is,

Any finite number multiplied by an infinitesimal number is equal to an infinitesimal number.

280. Interpretation of $\frac{a}{0}$.

The successive values of the fractions, $\frac{1}{2}, \frac{1}{.2}, \frac{1}{.02}, \frac{1}{.002}$, etc., are .5, 5, 50, 500, etc., and they continually increase as the denominators decrease.

In general, if the numerator of the fraction $\frac{a}{x}$ is constant while the denominator decreases regularly until it becomes numerically less than any assignable number, the quotient will increase regularly and become numerically *greater* than any assignable number.

$$\therefore \frac{a}{0} = \infty. \quad \text{That is,}$$

If a finite number is divided by an infinitesimal number, the quotient will be an infinite number.

281. Interpretation of $\frac{0}{0}$.

Let 0 represent absolute zero.

Then, if a is *any* finite number, § 279,

$$a \times 0 = 0;$$

whence,
$$\frac{0}{0} = a. \quad \text{That is,}$$

When 0 *represents absolute zero,* $\frac{0}{0}$ *is the symbol of an indeterminate number.*

282. Interpretation of $\frac{a}{\infty}$.

The successive values of the fractions, $\frac{1}{2}, \frac{1}{20}, \frac{1}{200}, \frac{1}{2000}$, etc., are .5, .05, .005, .0005, etc., and they continually decrease as the denominators increase.

In general, if the numerator of the fraction $\frac{a}{x}$ is constant while the denominator increases regularly until it becomes numerically greater than any assignable number, the quotient will decrease regularly and become numerically *less* than any

assignable number.

$$\therefore \frac{a}{\infty} = 0. \quad \text{That is,}$$

If a finite number is divided by an infinite number, the quotient will be an infinitesimal number.

283. Since (§ 280) $\frac{a}{0}$ is infinite and (§ 281) $\frac{0}{0}$ is indeterminate, it is seen that axiom 4 (§ 43) is not applicable when the divisor is 0; that is, *it is not allowable to divide by absolute zero.*

The student may point out the inadmissible step or fallacy in:

$$7x - 35 = 3x - 15,$$
$$7(x - 5) = 3(x - 5).$$
$$\therefore 7 = 3.$$

SUGGESTION. — Solve the equation to find what divisor has been used.

284. Fractions indeterminate in form.

Some fractions, for certain values of the variable involved, give the result $\frac{0}{0}$, which, however, is indeterminate *only in form*, because a definite value for the fraction may often be found.

For example, when $x = 1$, by substituting directly, $\frac{x^2 - 1}{x - 1} = \frac{0}{0}$.

Though $\frac{x^2 - 1}{x - 1} = \frac{(x + 1)(x - 1)}{x - 1} = x + 1$, it is not allowable to perform this operation in finding the value of the fraction when $x = 1$, that is, when $x - 1 = 0$, for (§ 283) it is not allowable to divide by absolute zero. However, since the value of $\frac{x^2 - 1}{x - 1}$ is always the same as the value of $x + 1$ so long as $x \neq 1$, let x approach 1 as a limit.

But (§ 274) x cannot become 1, and it is allowable to divide by $x - 1$.

Now as x approaches 1 as a limit, $\frac{x^2 - 1}{x - 1}$ approaches $x + 1$, or 2, as a limit, and so 2 is called the *value* of the fraction. That is,

The **value** of such a fraction for any given value of the variable involved is the *limit* that the fraction approaches as the variable approaches the given value as its limit.

PROGRESSIONS

285. A succession of numbers, each of which after the first is derived from the preceding number or numbers according to some fixed law, is called a **series**.

The successive numbers are called the **terms** of the series. The first and last terms are called the **extremes**, and all the others, the **means**.

In the series 2, 4, 6, 8, 10, 12, 14, each term after the first is greater by 2 than the preceding term. This is the **law** of the series. Also since 1st term $= 2 \cdot 1$, 2d term $= 2 \cdot 2$, 3d term $= 2 \cdot 3$, etc., the law of the series may be expressed thus:

$$n\text{th term} = 2\,n.$$

In the series 2, 4, 8, 16, 32, 64, 128, each term after the first is twice the preceding term; or expressing the law of the series by an equation, or formula,

$$n\text{th term} = 2^n.$$

ARITHMETICAL PROGRESSIONS

286. A series, each term of which after the first is derived from the preceding by the addition of a constant number, is called an **arithmetical series**, or an **arithmetical progression**.

The number that is added to any term to produce the next is called the **common difference**.

2, 4, 6, 8, $\cdots$ and 15, 12, 9, 6, $\cdots$ are arithmetical progressions. In the first, the common difference is 2 and the series is ascending; in the second, the common difference is -3 and the series is descending.

A.P. is an abbreviation of the words *arithmetical progression*.

287. To find the *n*th, or last, term of an arithmetical series.

In the arithmetical series

$$1, 3, 5, 7, 9, 11, 13, 15, 17, 19,$$

the common difference is 2, or $d=2$. This difference enters *once* in the *second* term, for $3=1+d$; *twice* in the *third* term, for $5=1+2d$; *three* times in the *fourth* term, for $7=1+3d$; and so on to the 10th, or last, term, which equals $1+9d$.

In $$a, a+d, a+2d, a+3d, \cdots,$$

which is the general form of an arithmetical progression, a representing the first term and d the common difference, observe that the coefficient of d in the expression for any term is *one less* than the number of the term.

Then, if the nth, or last, term is represented by l,

$$l=a+(n-1)d. \qquad \text{(I)}$$

NOTE. — The common difference d may be either positive or negative. In the A.P. 25, 23, 21, 19, 17, 15, $d=-2$.

EXERCISES

288. 1. What is the 10th term of the series 6, 9, 12, $\cdots$?

PROCESS

$$\begin{aligned} l &= a+(n-1)d \\ &= 6+(10-1)3 \\ &= 33 \end{aligned}$$

EXPLANATION. — Since the series 6, 9, 12, $\cdots$ is an A.P. the common difference of whose terms is 3, by substituting 6 for a, 10 for n, and 3 for d in the formula for the last term, the last term is found to be 33.

2. Find the 20th term of the series 7, 11, 15, $\cdots$. 83

3. Find the 16th term of the series 2, 7, 12, $\cdots$.

4. Find the 24th term of the series 1, 16, 31, $\cdots$.

5. Find the 18th term of the series 1, 8, 15, $\cdots$.

6. Find the 13th term of the series -3, 1, 5, $\cdots$.

7. Find the 49th term of the series 1, $1\frac{1}{3}$, $1\frac{2}{3}\cdots$.

8. Find the 15th term of the series 45, 43, 41, ⋯.

SUGGESTION. — The common difference is -2. 17

9. Find the 10th term of the series 5, 1, -3, ⋯.

10. Find the 16th term of the series a, $3a$, $5a$, ⋯.

11. Find the 7th term of the series $x-3y$, $x-2y$, ⋯.

12. A body falls $16\frac{1}{12}$ feet the first second, 3 times as far the second second, 5 times as far the third second, etc. How far will it fall during the 10th second?

Sum

289. To find the sum of *n* terms of an arithmetical series.

Let a represent the first term of an A.P., d the common difference, l the last term, n the number of terms, and s the sum of the terms.

Write the sum of n terms in the usual order and then in the reverse order, and add the two equal series; thus,

$$s = a + (a+d) + (a+2d) + (a+3d) + \cdots + l.$$

$$s = l + (l-d) + (l-2d) + (l-3d) + \cdots + a.$$

$$2s = (a+l) + (a+l) + (a+l) + (a+l) + \cdots + (a+l),$$

or $2s = n(a+l)$.

$$\therefore s = \frac{n}{2}(a+l), \text{ or } n\left(\frac{a+l}{2}\right). \qquad \text{(II)}$$

EXERCISES

290. 1. Find the sum of 20 terms of the series 2, 5, 8, ⋯.

PROCESS

$$l = a + (n-1)d = 2 + (20-1) \times 3 = 59$$

$$s = n\left(\frac{a+l}{2}\right) = 20\left(\frac{2+59}{2}\right) = 610$$

EXPLANATION. — Since the last term is not given, it is found by formula I and substituted for l in the formula for the sum.

Find the sum of:

2. 16 terms of the series 1, 5, 9, ⋯.

3. 10 terms of the series $-2, 0, 2, \cdots$.

4. 6 terms of the series $1, 3\frac{1}{2}, 6, \cdots$.

5. 8 terms of the series $a, 3a, 5a, \cdots$.

6. n terms of the series 1, 7, 13, ⋯.

7. a terms of the series $x, x+2a, \cdots$.

8. 7 terms of the series 4, 11, 18, ⋯.

9. 10 terms of the series $1, -1, -3, \cdots$.

10. 10 terms of the series $1, \frac{1}{2}, 0, \cdots$.

11. How many strokes does a common clock, striking hours, make in 12 hours?

12. A body falls $16\frac{1}{12}$ feet the first second, 3 times as far the second second, 5 times as far the third second, etc. How far will it fall in 10 seconds?

13. Thirty flowerpots are arranged in a straight line 4 feet apart. How far must a lady walk who, after watering each plant, returns to a well 4 feet from the first plant and in line with the plants, if we assume that she starts at the well?

14. How long is a toboggan slide, if it takes 12 seconds for a toboggan to reach the bottom by going 4 feet the first second and increasing its velocity 2 feet each second?

15. Starting from rest, a train went .18 of a foot the first second, .54 of a foot the next second, .90 of a foot the third second, and so on, reaching its highest speed in 3 minutes 40 seconds. How far did the train go before reaching top speed?

16. In a potato race each contestant has to start from a mark and bring back, one at a time, 8 potatoes, the first of which is 6 feet from the mark and each of the others 6 feet farther than the preceding. How far must each contestant go in order to finish the race?

291. The two fundamental formulæ,

$$\text{(I) } l = a + (n-1)d \text{ and (II) } s = \frac{n}{2}(a+l),$$

contain *five* **elements**, a, d, l, n, and s. Since these formulæ are independent simultaneous equations, if they contain but two unknown elements they may be solved. Hence, if any *three* of the five elements *are known*, the other *two* may be found.

EXERCISES

292. 1. Given $d = 3$, $l = 58$, $s = 260$, to find a and n.

SOLUTION

Substituting the known values in (I) and (II), we have

$$58 = a + (n-1)3, \text{ or } a + 3n = 61; \quad (1)$$

and $$260 = \tfrac{1}{2}n(a+58), \text{ or } an + 58n = 520. \quad (2)$$

Solving, we have $n = \frac{104}{3}$ or 5,

and, rejecting $n = \frac{104}{3}$, $a = 46$.

Since the number of terms must be a positive integer, fractional or negative values of n are rejected whenever they occur.

2. Given $a = 11$, $d = -2$, $s = 27$, to find the series.

SOLUTION

Substituting the known values in (I) and (II), we have

$$l = 11 + (n-1)(-2), \text{ or } l = 13 - 2n; \quad (1)$$

and $$27 = \tfrac{1}{2}n(11 + l), \text{ or } 54 = 11n + ln. \quad (2)$$

Solve, $n = 3$ or 9 and $l = 7$ or -5. (3)

Hence, the series is 11, 9, 7,

or 11, 9, 7, 5, 3, 1, -1, -3, -5.

3. How many terms are there in the series 2, 6, 10, $\cdots$, 66?

4. What is the sum of the series 1, 6, 11, $\cdots$, 61?

5. How many terms are there in the series $-1, 2, 5, \cdots$, if the sum is 221?

6. Find n and s in the series $2, 9, 16, \cdots, 86$.

7. Find l and s in $-10, -8\frac{1}{2}, -7, \cdots$ to 10 terms.

8. The sum of the series $\cdots, 22, 27, 32, \cdots$ is 714. If there are 17 terms, what are the first and last terms?

9. If $s = 113\frac{2}{3}$, $a = \frac{1}{3}$, and $d = 2$, find n.

10. What is the sum of the series $-16, -11, -6, \cdots, 34$?

11. What is the sum of the series $\cdots, -1, 3, 7, \cdots, 23$, if the number of terms is 16?

12. What are the extremes of the series $\cdots, 8, 10, 12, \cdots$, if $s = 300$, and $n = 20$?

13. Find an A.P. of 14 terms having 10 for its 6th term, 0 for its 11th term, and 98 for the sum of the terms.

14. Find an A.P. of 15 terms such that the sum of the 5th, 6th, and 7th terms is 60, and that of the last three terms, 132.

From (I) and (II) derive the formula for:

15. l in terms of a, n, s.

16. s in terms of a, d, l.

17. a in terms of d, n, s.

18. d in terms of a, n, s.

19. d in terms of l, n, s.

20. n in terms of a, l, s.

293. To insert arithmetical means.

EXERCISES

1. Insert 5 arithmetical means between 1 and 31.

SOLUTION

Since there are 5 means, there must be 7 terms. Hence, in $l = a + (n-1)d$, $l = 31$, $a = 1$, $n = 7$, and d is unknown.

Solving, we have $d = 5$.

Hence, 1, 6, 11, 16, 21, 26, 31 is the series.

2. Insert 9 arithmetical means between 1 and 6.

3. Insert 10 arithmetical means between 24 and 2.

4. Insert 7 arithmetical means between 10 and -14.

5. Insert 6 arithmetical means between -1 and 2.

6. Insert 14 arithmetical means between 15 and 20.

7. Insert 3 arithmetical means between $a-b$ and $a+b$.

294. If A is the arithmetical mean between a and b in the series

$$a,\ A,\ b,$$

by § 286,

$$A-a=b-A.$$

$$\therefore\ A=\frac{a+b}{2}.\quad \text{That is,}$$

PRINCIPLE. — *The arithmetical mean between two numbers is equal to half their sum.*

EXERCISES

295. Find the arithmetical mean between:

1. $\frac{2}{3}$ and $\frac{1}{2}$.

2. $a+b$ and $a-b$.

3. $(a+b)^2$ and $(a-b)^2$.

4. $\dfrac{x+y}{x-y}$ and $\dfrac{x-y}{x+y}$.

5. $1-x$ and $\dfrac{(1-x)^2}{1+x}$.

Problems

296. Problems in arithmetical progression involving two unknown elements commonly suggest series of the form,

$$x,\ x+y,\ x+2y,\ x+3y,\ \text{etc.}$$

Frequently, however, the solution of problems is more readily accomplished by representing the series as follows:

1. When there are *three* terms, the series may be written,

$$x-y,\ x,\ x+y.$$

2. When there are *five* terms, the series may be written,

$$x-2y,\ x-y.\ x,\ x+y,\ x+2y.$$

3. When there are *four* terms, the series may be written,

$$x - 3y,\ x - y,\ x + y,\ x + 3y.$$

The sum of the terms of a series represented as above evidently contains but one unknown number.

1. The sum of three numbers in arithmetical progression is 30 and the sum of their squares is 462. What are the numbers?

SOLUTION

Let the series be $x - y,\ x,\ x + y.$

Then, $$(x - y) + x + (x + y) = 30, \quad (1)$$

and $$(x - y)^2 + x^2 + (x + y)^2 = 462. \quad (2)$$

From (1), $$3x = 30; \quad (3)$$

whence, $$x = 10. \quad (4)$$

From (2), $$3x^2 + 2y^2 = 462. \quad (5)$$

Substitute (4) in (5), $$2y^2 = 162.$$

Solve, $$y = \pm 9.$$

Forming the series from $x = 10$ and $y = \pm 9$, we have for the terms

1, 10, 19 or 19, 10, 1.

2. The sum of three numbers in arithmetical progression is 18, and their product is 120. What are the numbers?

3. The sum of three numbers in arithmetical progression is 21, and the sum of their squares is 155. Find the numbers.

4. There are three numbers in arithmetical progression the sum of whose squares is 93. If the third is 4 times as large as the first, what are the numbers?

5. Find the sum of the odd numbers 1 to 99, inclusive.

6. The product of the extremes of an arithmetical progression of 10 terms is 70, and the sum of the series is 95. What are the extremes?

7. Fifty-five logs are to be piled so that the top layer shall consist of 1 log, the next layer of 2 logs, the next layer of 3 logs, etc. How many logs must be placed in the bottom layer?

8. It cost Mr. Smith \$19.00 to have a well dug. If the cost of digging was \$1.50 for the first yard, \$1.75 for the second, \$2.00 for the third, etc., how deep was the well?

9. How many arithmetical means must be inserted between 4 and 25, so that the sum of the series may be 116?

10. Prove that equal multiples of the terms of an arithmetical progression are in arithmetical progression.

11. Prove that the difference of the squares of consecutive integers are in arithmetical progression, and that the common difference is 2.

12. Prove that the sum of n consecutive odd integers, beginning with 1, is n^2.

GEOMETRICAL PROGRESSIONS

297. A series of numbers each of which after the first is derived by multiplying the preceding number by some constant multiplier is called a **geometrical series**, or a **geometrical progression.**

2, 4, 8, 16, 32 and a^4, a^3, a^2, a are geometrical progressions.
In the first series the constant multiplier is 2; in the second it is $\frac{1}{a}$.
G.P. is an abbreviation of the words *geometrical progression.*

298. The constant multiplier is called the **ratio.**

It is evident that the terms of a geometrical progression *increase* or *decrease* numerically according as the ratio is numerically *greater* or *less* than 1.

299. To find the *n*th, or last, term of a geometrical series.

Let a represent the first term of a G.P., r the ratio, n the number of terms, and l the last, or nth, term.

Then, the series is $a, ar, ar^2, ar^3, ar^4, \cdots$.

Observe that the exponent of r is one less than the number of the term; that is,

$$l = ar^{n-1}. \tag{I}$$

EXERCISES

300. 1. Find the 9th term of the series 1, 3, 9, ⋯.

PROCESS

$l = ar^{n-1}$

$= 1 \times 3^8$

$= 6561$

EXPLANATION. — In this exercise $a = 1$, $r = 3$, and $n = 9$.

Substituting these values in the formula for l, we have for the last term 6561.

2. Find the 10th term of the series 1, 2, 4, ⋯.

3. Find the 8th term of the series $\frac{1}{4}$, $\frac{1}{2}$, 1, ⋯.

4. Find the 9th term of the series 6, 12, 24, ⋯.

5. Find the 11th term of the series $\frac{1}{2}$, 1, 2, ⋯.

6. Find the 7th term of the series 2, 6, 18, ⋯.

7. Find the 6th term of the series 4, 20, 100, ⋯.

8. Find the 6th term of the series 6, 18, 54, ⋯.

9. Find the 10th term of the series 1, $\frac{1}{2}$, $\frac{1}{4}$, ⋯.

10. Find the 10th term of the series 1, $\frac{2}{3}$, $\frac{4}{9}$, ⋯.

11. Find the 8th term of the series $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{9}$, ⋯.

12. Find the 11th term of the series $a^{19}b$, $a^{18}b^2$, ⋯.

13. Find the nth term of the series 2, $\sqrt{2}$, 1, ⋯.

14. If a man begins business with a capital of \$2000 and doubles it every year for 6 years, how much is his capital at the end of the sixth year?

15. The population of the United States was 76.3 millions in 1900. If it doubles itself every 25 years, what will it be in the year 2000?

16. A man's salary was raised $\frac{1}{4}$ every year for 5 years. If his salary was \$512 the first year, what was it the sixth year?

17. The population of a city, which at a certain time was 20,736, increased in geometrical progression 25% each decade. What was the population at the end of 40 years?

18. A man who wanted 10 bushels of wheat thought \$1 a bushel too high a price; but he agreed to pay 2 cents for the first bushel, 4 cents for the second, 8 cents for the third, and so on. How much did the last bushel cost him?

19. The machinery in a manufacturing establishment is valued at \$20,000. If its value depreciates each year to the extent of 10% of its value at the beginning of that year, how much will the machinery be worth at the end of 5 years?

20. From a grain of corn there grew a stalk that produced an ear of 150 grains. These grains were planted, and each produced an ear of 150 grains. This process was repeated until there were 4 harvestings. If 75 ears of corn make 1 bushel, how many bushels were there the fourth year?

301. A series consisting of a limited number of terms is called a **finite series.**

302. A series consisting of an unlimited number of terms is called an **infinite series.**

303. To find the sum of a finite geometrical series. Sum.

Let a represent the first term, r the ratio, n the number of terms, l the nth, or last, term, and s the sum of the terms.

Then, $$s = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}. \qquad (1)$$

$(1) \times r,$ $$rs = ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n. \qquad (2)$$

$(2) - (1),$ $s(r-1) = ar^n - a.$

$$\therefore\ s = \frac{ar^n - a}{r-1}. \qquad \text{(II)}$$

But, since $ar^{n-1} = l,$ $\qquad ar^n = rl.$

Substituting rl for ar^n in (II), we have

$$s = \frac{rl - a}{r-1}, \text{ or } \frac{a - rl}{1 - r}. \qquad \text{(III)}$$

EXERCISES

304. 1. Find the sum of 6 terms of the series 3, 9, 27, ⋯.

PROCESS

$$s = \frac{ar^n - a}{r - 1}$$

$$= \frac{3 \times 3^6 - 3}{3 - 1} = 1092$$

EXPLANATION. — Since the first term a, the ratio r, and the number of terms n, are known, formula II, which gives the sum in terms of a, r, and n, is used.

2. Find the sum of 8 terms of the series 1, 2, 4, ⋯.

3. Find the sum of 8 terms of the series $1, \frac{1}{2}, \frac{1}{4}, \cdots$.

4. Find the sum of 10 terms of the series $1, 1\frac{1}{2}, 2\frac{1}{4}, \cdots$.

5. Find the sum of 7 terms of the series $2, -\frac{2}{3}, \frac{2}{9}, \cdots$.

6. Find the sum of 12 terms of the series $-\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \cdots$.

7. Find the sum of 7 terms of the series $1, 2x, 4x^2, \cdots$.

8. Find the sum of 7 terms of the series $1, -2x, 4x^2, \cdots$.

9. Find the sum of n terms of the series $1, x^2, x^4, \cdots$.

10. Find the sum of n terms of the series 1, 2, 4, ⋯.

11. Find the sum of n terms of the series $1, \frac{1}{3}, \frac{1}{9}, \cdots$.

12. The extremes of a geometrical series are 1 and 729, and the ratio is 3. What is the sum of the series?

13. What is the sum of the series 3, 6, 12, ⋯, 192?

14. What is the sum of the series $7, \cdots, -56, 112, -224$?

305. To find the sum of an infinite geometrical series.

If the ratio r is numerically less than 1, it is evident that the successive terms of a geometrical series become numerically less and less. Hence, in an infinite decreasing geometrical series, the nth term l, or ar^{n-1}, can be made less than any assignable number, though not absolutely equal to zero.

Formula (III), page 225, may be written,

$$s=\frac{a}{1-r}-\frac{rl}{1-r}.$$

Since by taking enough terms l and, consequently, rl can be made less than any assignable number, the second fraction may be neglected.

Hence, the formula for the sum of an infinite decreasing geometrical series is

$$s=\frac{a}{1-r}. \qquad \text{(IV)}$$

EXERCISES

306. 1. Find the sum of the series $1, \frac{1}{10}, \frac{1}{100}, \cdots$.

Solution

Substituting 1 for a and $\frac{1}{10}$ for r in (IV), we have

$$s=\frac{1}{1-\frac{1}{10}}=\frac{1}{\frac{9}{10}}=\frac{10}{9}.$$

Find the value of:

2. $1+\frac{1}{2}+\frac{1}{4}+\cdots$.

3. $3+\frac{3}{4}+\frac{3}{16}+\cdots$.

4. $1-\frac{1}{3}+\frac{1}{9}-\cdots$.

5. $-4-1-\frac{1}{4}-\cdots$.

6. $-2+\frac{2}{5}-\frac{2}{25}+\cdots$.

7. $100-10+1-\cdots$.

8. $1+x+x^2+x^3+\cdots$, when $x=.9$.

9. $1-x+x^2-x^3+\cdots$, when $x=\frac{3}{8}$.

10. Find the value of the repeating decimal $.185185185\cdots$.

Solution

Since $.185185185\cdots=.185+.000185+.000000185+\cdots$, $a=.185$ and $r=.001$.

Substitute in (IV), $\quad .185185185\cdots=s=\dfrac{.185}{1-.001}=\dfrac{5}{27}.$

Find the the value of:

11. $.407407\cdots$.

12. $.363636\cdots$.

13. $1.94444\cdots$.

14. $.020303\cdots$.

15. $.007007\cdots$.

16. $5.032828\cdots$.

307. **To insert geometrical means between two terms.**

EXERCISES

1. Insert 3 geometrical means between 2 and 162.

PROCESS

$l = ar^{n-1}$

$162 = 2r^4$

$r = \pm 3$

EXPLANATION. — Since there are three means, there are five terms, and $n-1=4$. Solving for r and neglecting imaginary values, we have $r = \pm 3$.

Therefore, the series is either 2, 6, 18, 54, 162 or 2, -6, 18, -54, 162.

2. Insert 3 geometrical means between 1 and 625.
3. Insert 5 geometrical means between $4\frac{1}{2}$ and $\frac{2048}{81}$.
4. Insert 4 geometrical means between $\frac{343}{16}$ and $\frac{64}{49}$.
5. Insert 4 geometrical means between 5120 and 5.
6. Insert 4 geometrical means between $4\sqrt{2}$ and 1.
7. Insert 5 geometrical means between a^6 and b^6.
8. Insert 4 geometrical means between x and $-y$.

308. If G is the geometrical mean between a and b, in the series

$$a,\ G,\ b,$$

by § 297,

$$\frac{G}{a} = \frac{b}{G}.$$

$$G = \pm\sqrt{ab}.$$ That is,

PRINCIPLE. — *The geometrical mean between two numbers is equal to the square root of their product.*

Observe that the *geometrical mean* between two numbers is also their *mean proportional.*

EXERCISES

309. Find the geometrical mean between:

1. 8 and 50.
2. $\frac{1}{2}$ and $3\frac{5}{9}$.
3. $1\frac{11}{16}$ and $\frac{3}{4}$.
4. $(a+b)^2$ and $(a-b)^2$.
5. $\dfrac{a^2+ab}{a^2-ab}$ and $\dfrac{ab+b^2}{ab-b^2}$.

310. Since formula I with formula II, or III, which is equivalent to II, forms a system of two independent simultaneous equations containing five elements, if *three* elements *are known*, the other *two* may be found by elimination.

NOTE. — Solving for n, since it is an exponent, requires a knowledge of logarithms (§§ 327–356), except in cases where its value may be determined by inspection. Only such cases are given in this chapter.

Problems

311. 1. Given r, l, and s, to find a.

2. The ratio of a geometrical progression is 5, the last term is 625, and the sum is 775. What is the first term?

3. The ratio of a geometrical progression is $\frac{1}{10}$, the sum is $\frac{1}{3}$, and the series is infinite. What is the first term?

4. Find l in terms of a, r, and s.

5. Find the last term of the series 5, 10, 20, $\cdots$, the sum of whose terms is 155.

6. If $\frac{1}{8}+\frac{1}{8}\sqrt{2}+\frac{1}{4}+\cdots=1\frac{7}{8}(1+\sqrt{2})$, what is the last term, and the number of terms?

7. Deduce the formula for r in terms of a, l, and s.

8. If the sum of the geometrical progression 32, $\cdots$, 243 is 665, what is the ratio? Write the series.

9. The sum of a geometrical progression is 700 greater than the first term and 525 greater than the last term. What is the ratio? If the first term is 81, what is the progression?

10. Deduce the formula for r in terms of a, n, and l.

11. The first term of a geometrical progression is 3, the last term is 729, and the number of terms is 6. What is the ratio? Write the series.

12. Find l in terms of r, n, and s.

13. A sled went 100 feet the first second after reaching the foot of a hill. How far did it go on the level, if its velocity decreased each second $\frac{1}{5}$ of that of the previous second?

14. Under normal conditions the members of a certain species of bacteria reproduce by division (each individual into two) every half hour. If no hindrance is offered, how many bacteria will a single individual produce in 8 hours?

15. A ball thrown vertically into the air 100 feet falls and rebounds 40 feet the first time, 16 feet the second time, and so on. What is the whole distance through which the ball will have passed when it finally comes to rest?

16. Show that the amount of $1 for 1, 2, 3, 4, 5 years at compound interest varies in geometrical progression.

17. Show that equal multiples of numbers in geometrical progression are also in geometrical progression.

18. The sum of three numbers in geometrical progression is 19, and the sum of their squares is 133. Find the numbers.

Suggestion. — When there are but three terms in the series, they may be represented by x^2, xy, y^2, or by x, $\sqrt{xy}$, y.

19. The product of three numbers in geometrical progression is 8, and the sum of their squares is 21. What are the three numbers?

20. The sum of the first and second of four numbers in geometrical progression is 15, and the sum of the third and fourth is 60. What are the numbers?

Suggestion. — Four unknown numbers in geometrical progression may be represented by $\frac{x^2}{y}$, x, y, $\frac{y^2}{x}$.

21. From a cask of vinegar $\frac{1}{3}$ was drawn off and the cask was filled by pouring in water. Show that if this is done 6 times, the contents of the cask will be more than $\frac{9}{10}$ water.

22. If the quantity, and correspondingly the pressure, of the air in the receiver of an air pump is diminished by $\frac{1}{10}$ of itself at each stroke of the piston, and if the initial pressure is 14.7 pounds per square inch, find, to the nearest tenth of a pound, what the pressure will be after 6 strokes.

GENERAL REVIEW

312. 1. Find the value of $2 + 6 \div 3 - 1 + 5 \times 2 + 7$. Explain the order in which the operations are performed.

2. Simplify $a-(b-c)-[a-\{b-c-(b+c-a)+(a-b)\}]$.

3. How does $kl \div yz$ differ in meaning from $k \times l \div y \times z$?

4. Show that $x(x - y + xy) = x^2 - xy + x^2y$, for as many numerical values as may be substituted for x and y.

5. Define known number; unknown number; positive number; negative number; like terms; coefficient; exponent.

6. Multiply $a^2b^3c^{-2}$ by $ab^{-2}c^4$; $-8\,ab$ by $2\,a^2b^3$; state the law of exponents for multiplication; the law of signs.

7. What laws are illustrated by $a(bc) = b(ac)$?

8. Divide $x^3y^{-4}z^5$ by x^2yz. State the **index** law for division.

9. When is $x^n - y^n$ divisible by $x - y$? by $x + y$?

10. When is $x^n + y^n$ divisible by $x - y$? by $x + y$?

11. Divide $x^4 - 4\,x + 5\,x^2 - 4\,x^3 + 1$ by $1 - 3\,x + x^2$, using detached coefficients.

12. Divide $x^5 - x + 2\,x^3 - 8 - 2\,x^4 + 12\,x^2$ by $x + 1$, using synthetic division.

13. In the expression $8\,x^6 + 4\,ax^5 - 2\,a^2x^2 - 3\,a^3x + a^5$, what is the degree of each term? What is the degree of the expression?

14. Distinguish between an *integral* and a *fractional* algebraic expression.

15. Solve graphically $\begin{cases} x + y = 8, \\ x - y = 2. \end{cases}$

16. During 12 hours of a certain day, the following temperatures were recorded: $-9°$, $-8°$, $-8°$, $-9°$, $-9°$, $-9°$, $-8°$, $+12°$, $+25°$, $+40°$, $+20°$, $+16°$. Find the average temperature for the 12 hours.

17. Reduce $\dfrac{6x^2 + ax - 12a^2}{3bx + 3xy - 4ab - 4ay}$ to lowest terms. When is a fraction in its lowest terms?

18. Show that $\dfrac{5}{9 - x^2} = \dfrac{-5}{x^2 - 9}$. Give the principles according to which the signs of the terms of a fraction may be changed.

19. Solve by two different methods $\begin{cases} 5x + y = 22, \\ x + 5y = 14. \end{cases}$

20. Represent the $\sqrt{8}$ by a line.

21. Find the fourth term of $\left(2x + \dfrac{1}{x}\right)^{10}$, when $x = 5$.

22. Define axiom; elimination; coördinate axes.

23. Construct the graph of $2y = 3x - 4$. Tell how to determine where a graph crosses the x-axis; the y-axis.

24. Show the difference in meaning between $(a^b)^c$ and $a^b \times a^c$.

25. Illustrate each of the following kinds of equations: numerical; literal; integral; fractional; identical; conditional; linear; homogeneous; symmetrical.

26. Solve the equation $2x^2 - 5x = 150$ by three methods. Explain each.

27. Define evolution; radical; surd; entire surd; mixed surd; binomial quadratic surd; similar surds; conjugate surds.

28. In the proportion $a : b = b : c$ indicate the extremes; the means; the mean proportional; the third proportional.

29. Illustrate how a root may be introduced in the solution of an equation; how a root may be removed. What is meant by an extraneous root?

30. Why is it specially important to test the values of the unknown number found in the solution of radical equations?

31. Upon what axiom is the clearing of equations of fractions based? What precautions should be taken to prevent introducing roots? If roots are introduced, how may they be detected?

32. Prove that a quadratic equation has two and only two roots.

33. Tell how to form a quadratic equation when its roots are given. Form the equation whose roots are $\frac{2}{3}$ and $\frac{1}{2}$.

34. What is the meaning of "function of x"? "infinite number"? Define variable.

35. Solve the equation $ax^2 + bx + c = 0$. Show the condition under which the roots are real and unequal; real and equal; imaginary; rational; surds; both positive; both negative; one positive and the other negative.

36. Derive the value of the sum of the roots of the equation $x^2 + px + q = 0$; the value of the product of the roots.

37. In clearing a fractional equation of its denominators, why should we multiply by their lowest common multiple?

Illustrate by showing what happens when the equation

$$\frac{2x}{x-1} - \frac{10}{x^2-1} = \frac{7}{x+1}$$

is multiplied by the product of all the denominators.

38. What powers of $\sqrt{-1}$ are real? imaginary?

39. Classify the following numbers as real or imaginary; as rational or irrational:

$$2,\ \sqrt{4},\ \sqrt{2},\ \sqrt{5},\ \sqrt{-2},\ \sqrt{-5},\ \sqrt{a^3},\ \sqrt[3]{a^6},\ \sqrt[4]{-a},$$

a being a positive number.

40. Find the ratio of $a^3 + b^3$ to $a^2 - ab + b^2$. Indicate the antecedent and the consequent in the ratio found.

41. Write the *inverse* ratio of a to b; the *duplicate* ratio.

42. For what values of x will $x^2 - x + 1 : x^2 + x + 1 = 3 : 7$?

43. If $a : b = c : d$, show that $2a + 3b : 2a = 2c + 3d : 2c$; $ma : nb = mc : nd$; $ma + nb : ma - nb = mc + nd : mc - nd$.

44. Write the formula for the sum s of an arithmetical series. Find the sum of 10 terms of the series 1, 4, 7, $\cdots$.

45. Prove that in a finite geometrical progression $s = \dfrac{a - rl}{1 - r}$.

46. Multiply $2x^{\frac{a}{2b}} - 5y^{\frac{a+b}{2}}$ by $2x^{\frac{a}{2b}} + 5y^{\frac{a+b}{2}}$.

47. Expand $(x^n - y^n)(x^n + y^n)(x^{2n} + y^{2n})$.

48. Divide $(a+b) + x$ by $(a+b)^{\frac{1}{3}} + x^{\frac{1}{3}}$.

49. Factor $9x^2 - 12x + 4$; $9x^2 + 9x + 2$; $x^3 - 3x + 2$; $a^4 + 1$.

50. Show by the factor theorem that $x - a$ is a factor of $x^n + 3ax^{n-1} - 4a^n$.

51. Separate $a^{12} - 1$ into six rational factors.

52. Factor $4(ad + bc)^2 - (a^2 - b^2 - c^2 + d^2)^2$.

53. Find the L.C.M. of $x^2 - y^2$, $x + y$, and $xy - y^2$.

54. Find the H.C.F. of $2x^4 - 7x^3 + 4x^2 + 7x - 6$,
$2x^4 + x^3 - 4x^2 + 7x - 15$, and $2x^4 + x^3 - x - 12$.

55. Expand $(2a + 3b)^4$; $(\sqrt{x} + \sqrt[3]{y})^6$; $(-1 - \sqrt{3})^3$.

If $a^m \times a^n = a^{m+n}$ for all values of m and n, show that:

56. $a^{-2} = \dfrac{1}{a^2}$.

57. $a^{\frac{3}{2}} = \sqrt{a^3} = (\sqrt{a})^3$.

58. $2a^{-\frac{1}{3}} = \dfrac{2\sqrt[3]{a^2}}{a}$.

59. $(ab)^0 = 1$.

60. $(abc)^3 = a^3b^3c^3$.

61. $\left(\dfrac{a}{b}\right)^3 = \dfrac{a^3}{b^3}$.

Simplify, expressing results with positive exponents:

62. $\dfrac{16^{\frac{3}{4}} \times 81^{\frac{1}{4}}}{25^{\frac{1}{2}} \times 64^{\frac{5}{6}}} \div 125^{-\frac{2}{3}}$.

63. $\left\{a^{-2}\left[a^{\frac{3}{5}}\left(a^{\frac{3}{5}}\right)^{\frac{1}{3}}\right]^5\right\}^{\frac{5}{2}}$.

64. $5^0 - \sqrt[5]{-32} + \sqrt[4]{256} - 8^{-\frac{2}{3}}$.

65. $\left[\dfrac{x^{-\frac{1}{2}}y^{-\frac{2}{3}}}{x^{-\frac{1}{6}}y^{-1}} \div \dfrac{x^{-2}y^2}{(xy)^{-3}}\right]^{-3}$.

66. $\left\{\left(a^{\frac{1}{2}}b^{\frac{5}{2}}\right)^{\frac{1}{5}} \div \left(a^{-\frac{1}{2}}b\right)^{-2}\right\}^{\frac{1}{9}}$.

67. $\dfrac{a+b}{a^{\frac{1}{3}} - b^{\frac{1}{3}}} - \dfrac{a-b}{a^{\frac{1}{3}} + b^{\frac{1}{3}}}$.

68. Find a factor that will rationalize $x^{\frac{2}{3}} + y^{\frac{3}{2}}$.

69 Interpret each of the following: $\dfrac{a}{0}$, $\dfrac{a}{\infty}$, $\dfrac{0}{0}$.

70. Simplify $\left[\dfrac{x}{1+\dfrac{1}{x}}+1-\dfrac{1}{x+1}\right] \div \left[\dfrac{x}{1-\dfrac{1}{x}}-x-\dfrac{1}{x-1}\right]$.

71. Simplify $\dfrac{1}{x-\dfrac{1}{x+\dfrac{1}{x}}}-\dfrac{1}{x+\dfrac{1}{x-\dfrac{1}{x}}}$.

Solve the following equations for x:

72. $mx^2 - nx = mn$.

73. $x^6 + 8 = 9x^3$.

74. $(1+x)^5 + (1-x)^5 = 242$.

75. $x + x^2 + (1 + x + x^2)^2 = 55$.

76. $\dfrac{1}{a-b}+\dfrac{a-b}{x}=\dfrac{1}{a+b}+\dfrac{a+b}{x}$.

77. $\dfrac{x-\dfrac{1}{a}}{c}+\dfrac{x-\dfrac{1}{b}}{a}+\dfrac{x-\dfrac{1}{c}}{b}=0$.

78. $\dfrac{1+x}{1+x+\sqrt{1+x^2}}=a-\dfrac{1+x}{1-x+\sqrt{1+x^2}}$.

Solve for x, y, and z:

79. $\begin{cases} x^4 + x^2y^2 + y^4 = 21, \\ x^2 - xy + y^2 = 7. \end{cases}$

80. $\begin{cases} ax + y + z = 2(a+1), \\ x + ay + z = 3a + 1, \\ x + y + az = a^2 + 3. \end{cases}$

81. $\begin{cases} x^2 + xy = -6, \\ y^2 + xy = 15. \end{cases}$

82. $\begin{cases} x^2 + 3xy = 7, \\ xy + 4y^2 = 18. \end{cases}$

83. $\begin{cases} x^2 + x = 26 - y^2 - y, \\ xy = 8. \end{cases}$

84. $\begin{cases} \sqrt{xy} = 12, \\ x + y - \sqrt{x+y} = 20. \end{cases}$

85. $\begin{cases} xy - xy^2 = -6, \\ x - xy^3 = 9. \end{cases}$

86. $\begin{cases} xy = x + y, \\ x^2 + y^2 = 8. \end{cases}$

87. $\begin{cases} x^2y^2 - 4xy = 5, \\ x^2 + 4y^2 = 29. \end{cases}$

88. $\begin{cases} 2x^3 + 2y^3 = 9xy, \\ x + y = 3. \end{cases}$

89. $\begin{cases} x^{\frac{1}{2}} + y^{\frac{1}{3}} = 4, \\ x^{\frac{3}{2}} + y = 16. \end{cases}$

Problems

313. 1. The sum of two numbers is 72 and their quotient is 8. Find the numbers.

2. The sum of $\frac{1}{5}$ and $\frac{1}{6}$ of a number multiplied by 4 equals 88. Find the number.

3. Separate 54 into two parts such that $\frac{1}{12}$ of the difference between them is $\frac{1}{3}$.

4. Separate m into two parts such that $\frac{1}{n}$ of the difference between them is $\frac{1}{r}$.

5. A man sold his crop of raisins for $480, thus gaining $\frac{13}{35}$ of the expense of raising them. What was this expense?

6. A rare book sold for $15,000. If there was a gain of $87\frac{1}{2}\%$, how much did the book cost?

7. From a rose farm 400,000 plants are sent out yearly. How many plants are there in a carload, if 25 times the number of cars is .001 of the number of plants in each?

8. A man who had no room for 8 of his horses, built an addition to his stable $\frac{1}{2}$ its size. He then had room for 8 horses more than he had. How many horses had he?

9. A woman on being asked how much she paid for eggs, replied, "Two dozen cost as many cents as I can buy eggs for 96 cents." What was the price per dozen?

10. The denominator of a certain fraction exceeds the numerator by 3. If both terms are increased by 4, the fraction will be increased by $\frac{1}{8}$. Find the fraction.

11. An expert workman makes 36,000 beads in a certain time. If he worked 2 days longer and made 1000 beads less per day, the total number of beads would be 40,000. How many beads does he make per day?

12. A dealer sold a number of horses for $1320, receiving the same price for each. If he had sold 1 horse less, but had charged $10 apiece more, he would have received the same sum. Find the price of a horse.

13. Two numbers are in the ratio of 7 to 9, but if 14 is added to each they will be in the ratio of 5 to 6. Find the numbers.

14. The meshed wire in a bundle had an area of 400 square feet. If its width had been 2 feet more, it would have been $\frac{1}{5}$ of its length. Find its dimensions.

15. The value of a fraction is $\frac{7}{8}$. If 4 is subtracted from its numerator and added to its denominator, the value of the resulting fraction is $\frac{3}{4}$. Find the fraction.

16. The greater of two numbers divided by the less gives a quotient of 7 and a remainder of 4; the less divided by the greater gives $\frac{2}{15}$. Find the numbers.

17. The greater of two numbers divided by the less gives a quotient of r and a remainder of s; the less divided by the greater gives t. Find the numbers.

18. There is a number whose three digits are the same. If 7 times the sum of the digits is subtracted from the number, the remainder is 180. What is the number?

19. A certain fraction plus its reciprocal equals $2\frac{1}{6}$. The numerator of the fraction minus the denominator equals 1. Find the fraction.

20. A firm finds that its monthly sales of toilet soap amount to \$40 more if put up 3 cakes to a box and sold for 12¢ a box, than if put up 6 cakes to a box and sold for 20¢ a box. How many cakes does the firm sell per month?

21. The perimeter of a rectangle is $8m$ and its area is $2m^2$. Find its dimensions.

22. The volumes of two cubes differ by 296 cubic inches and their edges differ by 2 inches. Find the edge of each.

23. The hypotenuse of a right triangle is 20. The sum of the other two sides is 28. Find the length of each side.

24. A sum of money at simple interest amounted in m years to a dollars and in n years to b dollars. Find the sum and the rate of interest.

25. A farmer sold a wagon for $16 and lost as many per cent as the number of dollars in the cost of the wagon. How much did the wagon cost?

26. If a number of two digits is divided by the product of its digits, the quotient is 6. If 9 is added to the number, the sum equals the number obtained by interchanging the digits. What is the number?

27. A piece of work can be done by A and B in 4 days, by A and C in 6 days, and by B and C in 12 days. Find the time it would take A to do it alone.

28. If the sides of an equilateral triangle are increased by 7 inches, 4 inches, and 1 inch, respectively, a right triangle is formed. Find the length of a side of the equilateral triangle.

29. A man planting peanuts with a machine used 30 pecks of shelled seed per day. If the number of acres planted per day was 1 more than the number of pecks of seed used per acre, how many acres of peanuts did he plant per day?

30. A jeweler has two silver cups. The first cup, with a cover on it valued at $1.50, is worth $1\frac{7}{8}$ times as much as the second cup, and the second cup with the cover on it is worth $\frac{11}{12}$ as much as the first cup. Find the value of each cup.

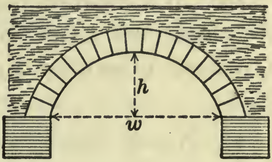
31. A merchant bought two lots of tea, paying for both $34. One lot was 20 pounds heavier than the other, and the number of cents paid per pound was in each case equal to the number of pounds bought. How many pounds of each did he buy?

32. Three farms for raising black foxes once contained together 75 foxes. The number of foxes on the three farms form a series in arithmetical progression, the largest number being 30. How many foxes were on each of the other farms?

33. Find two numbers such that their sum, their product, and the difference of their squares are all equal.

34. A tank contains 400 cubic feet. Its height exceeds its width by 1 foot and its length is 5 times its width. Find its dimensions.

35. A takes $1\frac{1}{2}$ hours longer than B to walk 15 miles, but if he doubles his rate he takes 1 hour less time than B. Find their rates of walking.



36. The height (h) of an arch of width (w) is given by the formula,

$$h = r - \sqrt{r^2 - (\tfrac{1}{2}w)^2},$$

in which r is the radius of the circle of which the arch is a segment. Solve for r.

37. The width of an arch for a culvert under a railroad embankment is 16 feet and its height is 6 feet. Find the radius of the arch.

38. How much does a teacher earn in 25 years, if she receives a salary of \$720 the first year, and an increase of \$80 each year for 14 years?

39. The sum of all the even integers from 2 to a certain number inclusive is 702. Find the last of these integers.

40. A and B can together do a piece of work in 15 days. After working together for 6 days, A went away, and B finished it 24 days later. In what time would A alone do the whole?

41. One machine makes 60 revolutions per minute more than another and in 5 minutes the former makes as many revolutions as the latter does in 8 minutes. Find the rate of each.

42. The area of the floor of a room is 120 square feet; of one end wall, 80 square feet; and of one side wall, 96 square feet. Find the dimensions of the room.

43. A company of soldiers attempted to form in a solid square, and 56 were left over. They attempted to form in a square with 3 more on each side, and there were 25 too few. How many soldiers were there in the company?

44. A tank can be filled by the larger of two faucets in 5 hours less time than by the smaller one. It is filled by them both together in 6 hours. How many hours will it take to fill the tank by each faucet separately?

45. How much pure alcohol must be added to a gallon of 92 % alcohol so that the mixture shall be 93 % alcohol?

46. In a mass of copper, lead, and tin, the copper weighed 5 pounds less than $\frac{1}{2}$ of the whole, and the lead and tin each 5 pounds more than $\frac{1}{3}$ of the remainder. Find the weight of each.

47. A new bronze was recently patented. It contained 5 % more copper than iron, twice as much nickel as aluminium, and 4 times the amount of aluminium was 3 % less than the amount of copper. What per cent of each metal did the bronze contain?

48. A needs 3 days more than B to do a certain piece of work, but working together the two men can do the work in 2 days. In how many days can B do the work?

49. Find three numbers in geometrical progression, such that their product is 1000, and the sum of the second and third is 6 times the first.

50. A rectangular field is 119 yards long and 19 yards wide. How many yards must be added to its width and how many taken from its length, in order that its area may remain the same, while its perimeter is increased by 24 yards?

51. It took a number of men as many days to pave a sidewalk as there were men. Had there been 3 men more, the work would have been done in 4 days. How many men were there?

52. By lowering the selling price of apples 2 cents a dozen, a man finds that he can sell 12 more than he used to sell for 60 cents. At what price per dozen did he sell them at first?

53. If the distance traveled by a train in 63 hours had been 44 miles less and its rate per hour had been $4\frac{2}{3}$ miles more, the trip would have taken 50 hours. Find the total run.

54. In a quantity of gunpowder the niter composed 10 pounds more than $\frac{3}{4}$ of the weight, the sulphur 3 pounds more than $\frac{1}{12}$ of it, and the charcoal 3 pounds less than $\frac{1}{10}$ of the weight of the niter. What was the weight of the gunpowder?

55. Two numbers whose product is 28,350, consist of three digits each. The hundreds' and units' digits of one are, respectively, 2 and 5, the corresponding digits of the other are 1 and 6, the tens' digit being the same in both numbers. Find the numbers.

56. If \$820 is put at interest for a certain number of years at a certain rate, it amounts to \$955.30. If the time were 1 year less and the rate $\frac{1}{2}\%$ more, the amount would be \$918.40. Find the time and the rate.

57. A and B can do a piece of work in m days, B and C in n days, A and C in p days. In what time can all together do it? How long will it take each alone to do it?

58. At \$2.50 per day, how many days did a man work to earn \$24, if he forfeited \$1.50 for every day he was idle, and worked 3 times as many days as he was idle?

59. A train, A, starts to go from P to Q, two stations 240 miles apart, and travels uniformly. An hour later another train, B, starts from P, and after traveling for 2 hours comes to a point that A passed 45 minutes previously. The rate of B is now increased by 5 miles an hour and it overtakes A just on entering Q. Find the rates at which the trains started.

60. It takes A and B $\frac{3}{5}$ of a day longer to tin and paint a roof than it does C and D, and the latter can do 50 square feet more a day than the former. If the roof contains 900 square feet, how much can A and B do in a day? C and D?

61. Find two numbers differing by 48, whose arithmetic mean exceeds the geometric mean by 18.

62. The formula for the weight of a hollow cylindrical column is

$$W = 3\pi lw(D^2 - d^2),$$

l being expressed in feet, W and w in pounds, and D and d in inches. Find the weight of a hollow cylindrical cast iron column in which $l = 10$, $w = .2607$ (pounds per cubic inch), the outside diameter $D = 8$, and the inside diameter $d = 4$.

63. A yacht goes 5 miles downstream in the same time that it goes 3 miles upstream; but if its rate each way is diminished 4 miles an hour, its rate downstream will be twice its rate upstream. How fast does it go in each direction?

64. On shipboard "eight bells" is rung at midnight and every 4 hours thereafter. If 1 bell is rung at 12.30 A.M. and the number of bells increases by 1 every half hour up to "eight bells," how many bells are rung in the 24 hours?

65. A man invested \$ 2720 in railroad stock, a part at 95 yielding 2 % and the balance at 82 yielding 3 %. His income from both investments was \$ 70. Find the amount invested in each kind of stock.

66. A rectangular piece of tin is 4 inches longer than it is wide. An open box containing 840 cubic inches is made from it by cutting a 6-inch square from each corner and turning up the ends and sides. What are the dimensions of the box?

67. A projectile fired from a battleship was heard by the gunner to strike a mark 3360 feet away $4\frac{1}{3}$ seconds after it was fired. An officer on another vessel 5600 feet from the first and 2240 feet from the mark heard the shot strike $1\frac{2}{3}$ seconds before the report reached him. Find the velocity of the sound and the average velocity of the projectile.

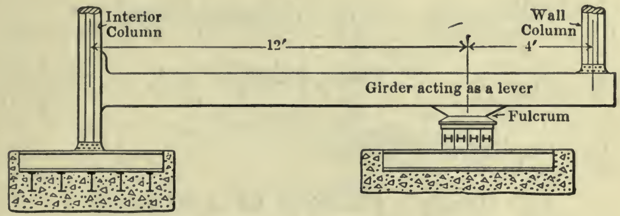
68. Find the common difference of the arithmetical progression whose first term is 3 and whose second, fourth, and eighth terms are in geometrical progression.

69. If zinc weighs 437.5 pounds per cubic foot and copper 550 pounds, what per cent by volume is each of these metals in an alloy of them, 1 cubic foot of which weighs 532 pounds?

70. The velocity acquired or lost by a body acted upon by gravity is given by the formula $v = gt$ (take $g = 32.16$). A bullet is fired vertically upward with an initial velocity of 2010 feet per second. Find in how many seconds it will return to the earth (neglecting the friction of the air).

Using the formula $s = \frac{1}{2} gt^2$, find how high the bullet will rise.

71. The load on a wall column for an office building is 360,000 pounds, including the weight of the column itself, and



is balanced, as shown in the figure, by a part of the load on an interior column. Neglecting the weight of the girder, find the load on the fulcrum.

72. A man bought some 50-dollar shares in one stock company and $\frac{2}{3}$ as many 100-dollar shares in another. At the end of the first quarter, dividends of 2 % and of $1\frac{1}{2}$ %, respectively, were declared on these stocks, and the man received \$ 120. How much money did he invest in each company?

73. It took a passenger train, 175 feet long, $7\frac{1}{2}$ seconds to pass completely a freight train, 485 feet long, moving in the opposite direction. If the passenger train was going 3 times as fast as the freight train, what was the rate of each per hour?

74. The distance a body will fall in t seconds, starting from rest, is given by the formula $s = \frac{1}{2}gt^2$. A man dropped a torpedo from a height and heard the report 5 seconds later. Taking $g = 32.16$ and the velocity of sound 1125.6 feet per second, find, to the nearest tenth of a second, the time during which the torpedo was falling.

75. A mixture of graphite and clay, to be used as "lead" in pencils, was c % clay and weighed p pounds. After the addition of clay to make the "lead" harder, the mixture was $(c + 10)$ % clay and weighed 240 pounds. If graphite had been added, instead of clay, until the mixture weighed 250 pounds, the mixture would have been $(c - 8)$ % clay. Solve for p and for c.

314. The following examination was given recently by the College Entrance Board for *Elementary Algebra Complete:*

1. (*a*) Factor
$2\,mx + 6\,ny - my - 12\,nx$; $6\,x^2 + 11\,x - 10$; $x^4 - a^3x + bx^3 - a^3b$.

(*b*) Simplify $1 - \left\{\dfrac{c^3 + y^3}{(c - y)^2} \div \left[\dfrac{c^4 + c^2y^2 + y^4}{c^3 - y^3} \times \dfrac{(c + y)^2}{c^2 - y^2}\right]\right\}$.

2. (*a*) Simplify and combine

$$\frac{6}{\sqrt{3}} - 18\sqrt{\tfrac{1}{3}} - \tfrac{1}{6}\sqrt{108} + 12^{\frac{1}{2}} + 3^{\frac{3}{2}} + \frac{\sqrt{3}}{5^{-1}}.$$

(*b*) Rationalize the denominator and simplify $\dfrac{\sqrt{3} - \sqrt{\frac{1}{2}}}{\sqrt{2} + \sqrt{\frac{1}{3}}}$.

3. (*a*) Solve $\begin{cases} 2\,x + y = -\,2, \\ 2\,xy - y^2 + 6\,x + 9 = 0. \end{cases}$

Associate properly the values of x and y.

(*b*) Solve $\sqrt{\dfrac{x}{2\,x + 1}} + 2\sqrt{\dfrac{2\,x + 1}{x}} = 3$.

4. (*a*) Solve $\begin{cases} \dfrac{1}{x^2} + \dfrac{1}{y^2} = 4\,c^2 + d^2, \\ xy = -\dfrac{1}{2\,cd}. \end{cases}$ Associate properly the values of x and y.

(*b*) If $b : c = 5 : 3$ in the equation $x^2 + bx + c^2 = 0$, are the roots of the equation real? Give the reason for your answer.

5. At his usual rate a man can row 15 miles downstream in 5 hours less than it takes him to return. Could he double his rate, his time downstream would be only 1 hour less than his time up. Find his rate in still water and the rate of the current.

6. The second term of an arithmetic progression is $\frac{1}{3}$ of the 8th and the sum of 20 terms is 63. Find the progression.

7. (*a*) Graph $y = 1 + 3\,x^2$.

(*b*) In the expansion of $\left(3\,x - \dfrac{1}{3\,x^{\frac{1}{2}}}\right)^7$ find the term which, when simplified, contains $x^{\frac{5}{2}}$.

SUPPLEMENTARY TOPICS

CUBE ROOT

Cube Root of Polynomials

EXERCISES

315. 1. Find the process for extracting the cube root of $a^3 + 3a^2b + 3ab^2 + b^3$.

PROCESS

$$\begin{array}{lll} & & a^3 + 3a^2b + 3ab^2 + b^3 \,\underline{|\, a + b} \\ & & \underline{a^3} \\ \text{Trial divisor,} & 3a^2 & \quad \big| 3a^2b + 3ab^2 + b^3 \\ \text{Complete divisor,} & 3a^2 + 3ab + b^2 & \quad \big| \underline{3a^2b + 3ab^2 + b^3} \end{array}$$

Explanation. — Since $a^3 + 3a^2b + 3ab^2 + b^3$ is the cube of $(a + b)$, we know that the cube root of $a^3 + 3a^2b + 3ab^2 + b^3$ is $a + b$.

Since the first term of the root is a, it may be found by taking the cube root of a^3, the first term of the power. On subtracting, there is a remainder of $3a^2b + 3ab^2 + b^3$.

The second term of the root is known to be b, and that may be found by dividing the first term of the remainder by 3 times the square of the part of the root already found. This divisor is called a *trial* divisor.

Since $3a^2b + 3ab^2 + b^3$ is equal to $b(3a^2 + 3ab + b^2)$, the complete divisor, which multiplied by b produces the remainder $3a^2b + 3ab^2 + b^3$, is $3a^2 + 3ab + b^2$; that is, the complete divisor is found by adding to the trial divisor 3 times the product of the first and second terms of the root and the square of the second term of the root.

On multiplying the complete divisor by the second term of the root, and on subtracting, there is no remainder; then, $a + b$ is the required root.

Since, in cubing $a + b + c$, $a + b$ may be expressed by x, the cube of the number will be $x^3 + 3x^2c + 3xc^2 + c^3$. Hence, it is obvious that the cube root of an expression whose root consists of *more than two terms* may be extracted in the same way as in exercise 1, *by considering the terms already found as one term.*

2. Find the cube root of $b^6 - 3b^5 + 5b^3 - 3b - 1$.

PROCESS

$$\begin{array}{lll|l}
 & & b^6 - 3b^5 + 5b^3 - 3b - 1 & \underline{b^2 - b - 1} \\
 & & b^6 & \\
\hline
\text{Trial divisor,} & 3b^4 & -3b^5 + 5b^3 & \\
\text{Complete divisor,} & 3b^4 - 3b^3 + b^2 & -3b^5 + 3b^4 - b^3 & \\
\hline
\text{Trial divisor,} & 3b^4 - 6b^3 + 3b^2 & -3b^4 + 6b^3 - 3b - 1 & \\
\text{Complete divisor,} & 3b^4 - 6b^3 + 3b + 1 & -3b^4 + 6b^3 - 3b - 1 & \\
\hline
\end{array}$$

EXPLANATION. — The first two terms are found in the same manner as in the previous exercise. In finding the next term, $b^2 - b$ is considered as one term, which we square and multiply by 3 for a trial divisor. On dividing the remainder by this trial divisor, the next term of the root is found to be -1. Adding to the trial divisor 3 times $(b^2 - b)$ multiplied by -1, and the square of -1, we obtain the complete divisor. On multiplying this by -1, and on subtracting the product from $-3b^4 + 6b^3 - 3b - 1$, there is no remainder. Hence, the cube root of the polynomial is $b^2 - b - 1$.

RULE. — *Arrange the polynomial with reference to the consecutive powers of some letter.*

Extract the cube root of the first term, write the result as the first term of the root, and subtract its cube from the given polynomial.

Divide the first term of the remainder by three times the square of the root already found, used as a trial divisor, and the quotient will be the next term of the root.

Add to this trial divisor three times the product of the first and second terms of the root, and the square of the second term. The result will be the complete divisor.

Multiply the complete divisor by the last term of the root found, and subtract this product from the dividend.

Find the next term of the root by dividing the first term of the remainder by the first term of the trial divisor.

Form the complete divisor as before, considering the part of the root already found as the first term, and continue in this manner until all the terms of the root are found.

Find the cube root of:

3. $b^3 + 6b^2 + 12b + 8.$

4. $x^3 - 3x^2y + 3xy^2 - y^3.$

5. $m^3 - 9m^2 + 27m - 27.$

6. $a^3x^6 + 12a^2x^4 + 48ax^2 + 64.$

7. $8m^3 - 60m^2n + 150mn^2 - 125n^3.$

8. $27x^3 - 189x^2y + 441xy^2 - 343y^3.$

9. $125a^3 + 675a^2x + 1215ax^2 + 729x^3.$

10. $64a^3b^3 - 240a^2b^2c + 300abc^2 - 125c^3.$

11. $x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1.$

12. $m^6 + 6m^5 + 15m^4 + 20m^3 + 15m^2 + 6m + 1.$

13. $x^6 + 12x^5 + 63x^4 + 184x^3 + 315x^2 + 300x + 125.$

14. $x^6 + 6x^5 - 18x^4 - 1000 + 180x^2 - 112x^3 + 600x.$

15. $8c^6 - 60c^5 + 198c^4 - 365c^3 + 396c^2 - 240c + 64.$

16. $c^6 - 3c^5d - 3c^4d^2 + 11c^3d^3 + 6c^2d^4 - 12cd^5 - 8d^6.$

17. $x^3 - 12x^2 + 54x - 112 + \dfrac{108}{x} - \dfrac{48}{x^2} + \dfrac{8}{x^3}.$

18. $\dfrac{a^3b^3x^9}{c^3} - \dfrac{c^3x^6}{b^3} + \dfrac{3acx^7}{b} - \dfrac{3a^2bx^8}{c}.$

19. $x^6 + 15x^2 + \dfrac{15}{x^2} + 20 + \dfrac{6}{x^4} + \dfrac{1}{x^6} + 6x^4.$

20. $\dfrac{1}{x^3} - \dfrac{3}{2x^2} + \dfrac{27}{4x} - \dfrac{49}{8} + \dfrac{27x}{2} - 6x^2 + 8x^3.$

21. $n^6 - \frac{3}{2}n^5 + \frac{9}{4}n^4 - \frac{13}{8}n^3 + \frac{9}{8}n^2 - \frac{3}{8}n + \frac{1}{8}.$

22. $\frac{1}{8}x^6 + \frac{3}{4}x^5y + x^4y^2 - x^3y^3 - \frac{4}{3}x^2y^4 + \frac{4}{3}xy^5 - \frac{8}{27}y^6.$

Cube Root of Arithmetical Numbers

316. Compare the number of digits in the cube root of each number with the number of digits in the number itself:

NUMBER	ROOT	NUMBER	ROOT	NUMBER	ROOT
1	1	1′000	10	1′000′000	100
27	3	27′000	30	27′000′000	300
729	9	970′299	99	997′002′999	999

Observe that:

PRINCIPLE. — *If a number is separated into periods of three digits each, beginning at units, its cube root will have as many digits as the number has periods.*

The left-hand period may be incomplete, consisting of only one or two digits.

317. If the number of units expressed by the tens' digit is represented by t, and the number of units expressed by the units' digit is represented by u, any number consisting of tens and units may be represented by $t + u$, and its cube by $(t + u)^3$, or $t^3 + 3\,t^2u + 3\,tu^2 + u^3$.

Thus, $\quad 25 = 2$ tens $+ 5$ units, or $(20 + 5)$ units,

and $\quad 25^3 = 20^3 + 3(20^2 \times 5) + 3(20 \times 5^2) + 5^3 = 15{,}625.$

EXERCISES

318. **1.** Extract the cube root of 12,167.

FIRST PROCESS

$$
\begin{array}{lrr|l}
 & & 12'167 & \underline{20 + 3} \\
 & t^3 = & 8\,000 & \\
\hline
\text{Trial divisor,} & 3\,t^2 = 1200 & 4\,167 & \\
 & 3\,tu = 180 & & \\
 & u^2 = 9 & & \\
\hline
\text{Complete divisor,} & = 1389 & 4\,167 &
\end{array}
$$

EXPLANATION. — On separating 12,167 into periods of three figures each (§ 316), there are found to be two digits in the root, that is, the root is composed of *tens* and *units*. Since the cube of tens is thousands, and the thousands of the power are less than 27, or 3^3, and more than 8, or 2^3, the tens' figure of the root is 2. 2 tens, or 20, cubed is 8000, and 8000 sub-

tracted from 12,167 leaves 4167, which is equal to 3 times the tens^2 $\times$ the units + 3 times the tens $\times$ the units^2 + the units^3.

Since 3 times the tens^2 $\times$ the units is much greater than 3 times the tens $\times$ the units^2 + the units^3, 4167 is only a little more than 3 times the tens^2 $\times$ the units. If, then, 4167 is divided by 3 times the tens^2, or by 1200, the trial divisor, the quotient will be approximately equal to the units, that is, 3 will be the units of the root, provided proper allowance has been made for the additions necessary to obtain the complete divisor.

Since the complete divisor is found by adding to 3 times the tens^2 the sum of 3 times the tens $\times$ the units and the units^2, the complete divisor is $1200 + 180 + 9$, or 1389. This multiplied by 3, the units, gives 4167, which, subtracted from 4167, leaves no remainder.

Therefore, the cube root of 12,167 is $20 + 3$, or 23.

SECOND PROCESS

$$
\begin{array}{rr|l}
 & 12'167 & \underline{23} \\
t^3 = & \dot{8} & \\
\hline
3t^2 = 1200 & 4\,167 & \\
3tu = 180 & & \\
u^2 = 9 & & \\
\hline
1389 & 4\,167 &
\end{array}
$$

EXPLANATION. — In practice it is usual to place figures of the same order in the same column, and to disregard the ciphers on the right of the products.

Since a root expressed by any number of figures may be regarded as composed of tens and units, the processes of exercise 1 have a general application.

Thus, $120 = 12$ tens + 0 units ; $1203 = 120$ tens + 3 units.

2. Extract the cube root of 1,740,992,427.

SOLUTION

$$
\begin{array}{llr|l}
 & & & 1'740'992'427 \;\underline{|1203} \\
 & t^3 = & & 1 \\
\hline
\text{Complete divisor} & 3t^2 = 3(10)^2 & = 300 & 740 \\
 & 3tu = 3(10 \times 2) & = 60 & \\
 & u^2 = 2^2 & = 4 & \\
\hline
 & & 364 & 728 \\
\hline
 & & & 12\,992 \\
 & 3t^2 = 3(120)^2 & = 43200 & \\
\hline
\text{Complete divisor} & 3t^2 = 3(1200)^2 & = 4320000 & 12\,992\,427 \\
 & 3tu = 3(1200 \times 3) & = 10800 & \\
 & u^2 = 3^2 & = 9 & \\
\hline
 & & 4330809 & 12\,992\,427
\end{array}
$$

Since the third figure of the root is 0, it is not necessary to form the complete divisor, inasmuch as the product to be subtracted will be 0.

RULE. — *Separate the number into periods of three figures each, beginning at units. Find the greatest cube in the left-hand period, and write its root for the first digit of the required root.*

Cube this root, subtract the result from the left-hand period, and annex to the remainder the next period for a new dividend.

Take three times the square of the root already found, annex two ciphers for a trial divisor, and by the result divide the dividend. The quotient, or the quotient diminished, will be the second figure of the root.

To this trial divisor add three times the product of the first part of the root with a cipher annexed, multiplied by the second part, and also the square of the second part Their sum will be the complete divisor.

Multiply the complete divisor by the second part of the root, and subtract the product from the dividend.

Continue thus until all the figures of the root have been found.

1. When there is a remainder after subtracting the last product, annex decimal ciphers, and continue the process.

2. Decimals are pointed off from the decimal point toward the right.

3. The cube root of a common fraction may be found by extracting the cube root of the numerator and the denominator separately or by reducing the fraction to a decimal and then extracting its root.

Extract the cube root of:

3. 29,791.	9. 2,406,104.	15. .000024389.
4. 54,872.	10. 69,426,531.	16. .001906624.
5. 110,592.	11. 28,372,625.	17. .000912673.
6. 300,763.	12. 48.228544.	18. .259694072.
7. 681,472.	13. 17,173.512.	19. 926.859375.
8. 941,192.	14. 95.443993.	20. 514,500.058197.

Extract the cube root to three decimal places:

21. 2.	23. .8.	25. $\frac{5}{64}$.	27. $\frac{7}{8}$.
22. 5.	24. .16.	26. $\frac{2}{3}$.	28. $\frac{3}{16}$.

VARIATION

319. One quantity or number is said to **vary directly** as another, or simply to **vary** as another, when the two depend upon each other in such a manner that if one is changed the other is changed *in the same ratio*.

Thus, if a man earns a certain sum per day, the amount of wages he earns *varies* as the number of days he works.

320. The **sign of variation** is $\propto$. It is read '*varies as.*'

Thus, $x \propto y$, read 'x varies as y,' is a brief way of writing the proportion

$$x : x' = y : y',$$

in which x' is the value to which x is changed when y is changed to y'.

321. The expression $x \propto y$ means that if y is doubled, x is doubled, or if y is divided by a number, x is divided by the same number, etc.; that is, that the ratio of x to y is always the same, or *constant*. If the constant ratio is represented by k, then when $x \propto y$, $\frac{x}{y} = k$, or $x = ky$. Hence,

If x varies as y, x is equal to y multiplied by a constant.

322. One quantity or number varies **inversely** as another when it varies as the *reciprocal* of the *other*.

Thus, the time required to do a certain piece of work varies *inversely* as the number of men employed. For, if it takes 10 men 4 days to do a piece of work, it will take 5 men 8 days, or 1 man 40 days, to do it.

In $x \propto \frac{1}{y}$, if the constant ratio of x to $\frac{1}{y}$ is k, $\frac{x}{\frac{1}{y}} = k$, or $xy = k$.

Hence,

If x varies inversely as y, their product is a constant.

323. One quantity or number varies **jointly** as two others when it varies as their product.

Thus, the amount of money a man earns varies *jointly* as the number of days he works and the sum he receives per day. For, if he should work *three* times as many days, and receive *twice* as many dollars per day, he would receive *six* times as much money.

In $x \propto yz$, if the constant ratio of x to yz is k,

$$\frac{x}{yz} = k, \text{ or } x = kyz. \quad \text{Hence,}$$

If x varies jointly as y and z, x is equal to their product multiplied by a constant.

324. One quantity or number varies **directly** as a second and **inversely** as a third when it varies *jointly* as the second and the reciprocal of the third.

Thus, the time required to dig a ditch varies *directly* as the length of the ditch and *inversely* as the number of men employed. For, if the ditch were 10 times as long and 5 times as many men were employed, it would take twice as long to dig it.

In $x \propto y \cdot \frac{1}{z}$, or $x \propto \frac{y}{z}$, if k is the constant ratio,

$$x \div \frac{y}{z} = k, \text{ or } x = k\frac{y}{z}. \quad \text{Hence,}$$

If x varies directly as y and inversely as z, x is equal to $\frac{y}{z}$ multiplied by a constant.

325. *If x varies as y when z is constant, and x varies as z when y is constant, then x varies as yz when both y and z are variable.*

Thus, the area of a triangle varies as the base when the altitude is constant; as the altitude when the base is constant; and as the product of the base and the altitude when both vary.

Proof. — Since the variation of x depends upon the variations of y and z, suppose the latter variations to take place in succession, each in turn producing a corresponding variation in x.

While z remains constant, let y change to y_1, thus causing x to change to x'.

Then, $$\frac{x}{x'} = \frac{y}{y_1}. \tag{1}$$

Now while y keeps the value y_1, let z change to z_1, thus causing x' to change to x_1.

Then, $$\frac{x'}{x_1} = \frac{z}{z_1}. \tag{2}$$

Multiply (1) by (2), $\frac{x}{x_1}=\frac{yz}{y_1z_1}$. (3)

$$x=\frac{x_1}{y_1z_1}\cdot yz. \quad (4)$$

Since, if *both* changes are made, x_1, y_1, and z_1 are constants, $\frac{x_1}{y_1z_1}$ is a constant, which may be represented by k.

Then, (4) becomes $x = kyz$.

Hence, $x \propto yz$.

Similarly, if x varies as each of three or more numbers, y, z, v, $\cdots$ when the others are constant, when all vary x varies as their product.

That is, $x \propto yzv \cdots$.

Thus, the volume of a rectangular solid varies as the length, if the width and thickness are constant; as the width, if the length and thickness are constant; as the thickness, if the length and width are constant; as the product of any two dimensions, if the other dimension is constant; and as the product of the three dimensions, if all vary.

EXERCISES

326. 1. If x varies inversely as y, and $x=6$ when $y=8$, what is the value of x when $y=12$?

SOLUTION. — Since $x \propto \frac{1}{y}$, let k be the constant ratio of x to $\frac{1}{y}$.

Then, § 322, $xy = k$. (1)

Hence, when $x = 6$ and $y = 8$, $k = 6 \times 8$, or 48. (2)

Since k is constant, $k = 48$ when $y = 12$,

and (1) becomes $12x = 48$.

Therefore, when $y = 12$, $x = 4$.

2. If $x \propto \frac{y}{z}$, and if $x = 2$ when $y = 12$ and $z = 2$, what is the value of x when $y = 84$ and $z = 7$?

3. If $x \propto \frac{y}{z}$, and if $x = 60$ when $y = 24$ and $z = 2$, what is the value of y when $x = 20$ and $z = 7$?

4. If x varies jointly as y and z and inversely as the square of w, and if $x = 30$ when $y = 3$, $z = 5$, and $w = 4$, what is the value of x expressed in terms of y, z, and w?

5. If $x \propto y$ and $y \propto z$, prove that $x \propto z$.

Proof. — Since $x \propto y$ and $y \propto z$, let m represent the constant ratio of x to y, and n the constant ratio of y to z.

Then, § 321, $\quad x = my, \qquad (1)$

and $\quad y = nz. \qquad (2)$

Substitute nz for y in (1), $\quad x = mnz. \qquad (3)$

Hence, since mn is constant, $\quad x \propto z.$

6. If $x \propto \frac{1}{y}$ and $y \propto \frac{1}{z}$, prove that $x \propto z$.

7. If $x \propto y$ and $z \propto y$, prove that $(x \pm z) \propto y$.

8. The volume of a cone varies jointly as its altitude and the square of the diameter of its base. When the altitude is 15 and the diameter of the base is 10, the volume is 392.7. What is the volume when the altitude is 5 and the diameter of the base is 20?

Solution. — Let V, H, and D denote the volume, altitude, and diameter of the base, respectively, of any cone, and V' the volume of a cone whose altitude is 5 and the diameter of whose base is 20.

Since $\quad V \propto HD^2$, or $V = kHD^2$,

and $\quad V = 392.7$ when $H = 15$ and $D = 10$,

$$392.7 = k \times 15 \times 100. \qquad (1)$$

Also, since V becomes V' when $H = 5$ and $D = 20$,

$$V' = k \times 5 \times 400. \qquad (2)$$

Dividing (2) by (1), Ax. 4, $\quad \frac{V'}{392.7} = \frac{5 \times 400}{15 \times 100} = \frac{4}{3}. \qquad (3)$

$$\therefore V' = \tfrac{4}{3} \text{ of } 392.7 = 523.6.$$

9. The circumference of a circle varies as its diameter. If the circumference of a circle whose diameter is 1 foot is 3.1416 feet, what is the circumference of a circle 100 feet in diameter?

10. The area of a circle varies as the square of its diameter. If the area of a circle whose diameter is 10 feet is 78.54 square feet, what is the area of a circle whose diameter is 20 feet?

11. The distance a body falls from rest varies as the square of the time of falling. If a stone falls 64.32 feet in 2 seconds, how far will it fall in 5 seconds?

12. The volume of a sphere varies as the cube of its diameter. If the ratio of the sun's diameter to the earth's is 109.3, how many times the volume of the earth is the volume of the sun?

13. If 10 men can do a piece of work in 20 days, how long will it take 25 men to do it?

14. If a men can do a piece of work in b days, how many men will be required to do it in c days?

15. The illumination from a source of light varies inversely as the square of the distance. How far must a screen that is 10 feet from a lamp be moved so as to receive $\frac{1}{4}$ as much light?

16. The number of times a pendulum oscillates in a given time varies inversely as the square root of its length. If a pendulum 39.1 inches long oscillates once a second, what is the length of a pendulum that oscillates twice a second?

17. Three spheres of lead whose radii are 6 inches, 8 inches, and 10 inches, respectively, are united into one. What is the radius of the resulting sphere, if the volume of a sphere varies as the cube of its radius?

18. A wrought-iron bar 1 square inch in cross section and 1 yard long weighs 10 pounds. If the weight of a uniform bar of given material varies jointly as its length and the area of its cross section, what is the weight of a wrought-iron bar 36 feet long, 4 inches wide, and 4 inches thick?

19. The distances, from the fulcrum of a lever, of two weights that balance each other vary inversely as the weights. If two boys weighing 80 pounds and 90 pounds, respectively, are balanced on the ends of a board $8\frac{1}{2}$ feet long, how much of the board has each on his side of the fulcrum?

20. The weight of wire of given material varies jointly as the length and the square of the diameter. If 3 miles of wire .08 of an inch in diameter weigh 288 pounds, what is the weight of $\frac{1}{2}$ of a mile of wire .16 of an inch in diameter?

LOGARITHMS

327. The exponent of the power to which a fixed number, called the **base**, must be raised in order to produce a given number is called the **logarithm** of the given number.

When 2 is the base, the logarithm of 8 is 3, for $8 = 2^3$.

328. When a is the base, x the exponent, and m the given number, that is, when $a^x = m$, x is the logarithm of the number m to the base a, written $\log_a m = x$.

When the base is 10, it is not indicated. Thus, the logarithm of 100 to the base 10 is 2, and of 1000, 3; written, $\log 100 = 2$; $\log 1000 = 3$.

329. Logarithms may be computed with any arithmetical number except 1 as a base, but the base of the **common,** or **Briggs, system** of logarithms is 10.

Since $10^0 = 1$, the logarithm of 1 is 0.
Since $10^1 = 10$, the logarithm of 10 is 1.
Since $10^2 = 100$, the logarithm of 100 is 2.
Since $10^{-1} = \frac{1}{10}$, the logarithm of .1 is -1.
Since $10^{-2} = \frac{1}{100}$, the logarithm of .01 is -2.

330. Then, the logarithm of any number between 1 and 10 is greater than 0 and less than 1, and that of any number between 10 and 100 is greater than 1 and less than 2.

For example, the logarithm of 4 is approximately 0.6021, and of 50, approximately 1.6990. Most logarithms are endless decimals.

331. The integral part of a logarithm is called the **characteristic**; the fractional or decimal part, the **mantissa.**

In $\log 50 = 1.6990$, the characteristic is 1 and the mantissa is .6990.

332. The following illustrate *characteristics* and *mantissas:*

$\log 4580 = 3.6609$; that is, $4580 = 10^{3.6609}$.
$\log 458.0 = 2.6609$; that is, $458.0 = 10^{2.6609}$.
$\log 45.80 = 1.6609$; that is, $45.80 = 10^{1.6609}$.
$\log 4.580 = 0.6609$; that is, $4.580 = 10^{0.6609}$.
$\log .4580 = \bar{1}.6609$; that is, $.4580 = 10^{-1+.6609}$.
$\log .0458 = \bar{2}.6609$; that is, $.0458 = 10^{-2+.6609}$.
$\log .00458 = \bar{3}.6609$; that is, $.00458 = 10^{-3+.6609}$.

333. From the preceding examples it is evident that:

PRINCIPLES. — 1. *The characteristic of the logarithm of a number greater than* 1 *is either positive or zero and* 1 *less than the number of digits in the integral part of the number.*

2. *The characteristic of the logarithm of a decimal is negative and numerically* 1 *greater than the number of ciphers immediately following the decimal point.*

334. To avoid writing a negative characteristic before a positive mantissa, it is customary to add 10 or some multiple of 10 to the negative characteristic, and to indicate that the number added is to be subtracted from the whole logarithm.

Thus, $\bar{1}.6609$ is written $9.6609 - 10$; $\bar{2}.3010$ is written $8.3010 - 10$ or sometimes $18.3010 - 20$; $28.3010 - 30$; etc.

335. It is evident, also, from the examples in § 332, that in the logarithms of numbers expressed by the same figures in the same order, the decimal parts, or *mantissas*, are the same, and the logarithms differ only in their *characteristics*. Hence, tables of logarithms contain only the *mantissas*.

336. The table of logarithms on the two following pages gives the mantissas, to the nearest fourth place, of the common logarithms of all numbers from 1 to 1000.

337. To find the logarithm of a number.

EXERCISES

1. Find the logarithm of 765.

SOLUTION. — In the following table, the letter **N** designates a vertical column of numbers from **10** to **99** inclusive, and also a horizontal row of figures **0, 1, 2, 3, 4, 5, 6, 7, 8, 9**. The first two figures of 765 appear as the number **76** in the vertical column marked **N** on page 259, and the third figure **5** in the horizontal row marked **N**. In the same horizontal row as **76** are found the mantissas of the logarithms of the numbers 760, 761, 762, 763, 764, 765, etc. The mantissa of the logarithm of 765 is found in this row under **5**, the third figure of 765. It is 8837 and means .8837.

By Prin. 1, the characteristic of the logarithm of 765 is 2.

Hence, the logarithm of 765 is 2.8837.

Table of Common Logarithms

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
N	0	1	2	3	4	5	6	7	8	9

Table of Common Logarithms

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
N	0	1	2	3	4	5	6	7	8	9

2. Find the logarithm of 4.

Solution. — Although the numbers in the table appear to begin with 100, the table really includes all numbers from 1 to 1000, since numbers expressed by less than three figures may be expressed by three figures by adding decimal ciphers. Since $4 = 4.00$, and since, § 335, the mantissa of the logarithm of 4.00 is the same as that of 400, which is .6021, the mantissa of the logarithm of 4 is .6021.

By Prin. 1, the characteristic of the logarithm of 4 is 0.

Therefore, the logarithm of 4 is 0.6021.

Verify the following from the table:

3. $\log 10 = 1.0000$.
4. $\log 100 = 2.0000$.
5. $\log 110 = 2.0414$.
6. $\log 2 = 0.3010$.
7. $\log 20 = 1.3010$.
8. $\log 200 = 2.3010$.
9. $\log .2 = 9.3010 - 10$.
10. $\log 542 = 2.7340$.
11. $\log 345 = 2.5378$.
12. $\log 5.07 = 0.7050$.
13. $\log 78.5 = 1.8949$.
14. $\log .981 = 9.9917 - 10$.

15. Find the logarithm of 6253.

Solution. — Since the table contains the mantissas not only of the logarithms of numbers expressed by three figures, but also of logarithms expressed by four figures when the last figure is 0, the mantissa of the logarithm of 625 is first found, since, § 335, it is the same as the mantissa of the logarithm of 6250. It is found to be .7959.

The next greater mantissa is .7966, the mantissa of the logarithm of 6260. Since the numbers 6250 and 6260 differ by 10, and the mantissas of their logarithms differ by 7 ten-thousandths, it may be assumed as sufficiently accurate that each increase of 1 unit, as 6250 increases to 6260, produces a corresponding increase of .1 of 7 ten-thousandths in the mantissa of the logarithm. Consequently, 3 added to 6250 will add .3 of 7 ten-thousandths, or 2 ten-thousandths, to the mantissa of the logarithm of 6250 for the mantissa of the logarithm of 6253.

Hence, the mantissa of the logarithm of 6253 is $.7959 + .0002$, or .7961.

Since 6253 is an integer of 4 digits, the characteristic is 3 (Prin. 1).

Therefore, the logarithm of 6253 is 3.7961.

Note. — The difference between two successive mantissas in the table is called the *tabular difference*.

Find the logarithm of:

16. 1054.	20. 21.09.	24. .09095.
17. 1272.	21. 3.060.	25. .10125.
18. .0165.	22. 441.1.	26. 54.675.
19. 1906.	23. .7854.	27. .09885.

338. To find a number whose logarithm is given.

The number that corresponds to a given logarithm is called its **antilogarithm**.

Thus, since the logarithm of 62 is 1.7924, the antilogarithm of 1.7924 is 62.

EXERCISES

339. 1. Find the number whose logarithm is 0.9472.

SOLUTION. — The two mantissas adjacent to the given mantissa are .9469 and .9474, corresponding to the numbers 8.85 and 8.86, since the given characteristic is 0. The given mantissa is 3 ten-thousandths greater than the mantissa of the logarithm of 8.85, and the mantissa of the logarithm of 8.86 is 5 ten-thousandths greater than that of the logarithm of 8.85.

Since the numbers 8.85 and 8.86 differ by 1 one-hundredth, and the mantissas of their logarithms differ by 5 ten-thousandths, it may be assumed as sufficiently accurate that each increase of 1 ten-thousandth in the mantissa is produced by an increase of $\frac{1}{5}$ of 1 one-hundredth in the number. Consequently, an increase of 3 ten-thousandths in the mantissa is produced by an increase of $\frac{3}{5}$ of 1 one-hundredth, or .006, in the number.

Hence, the number whose logarithm is 0.9472 is 8.856.

2. Find the antilogarithm of $9.4180 - 10$.

SOLUTION. — Given mantissa,	.4180	
Mantissa next less,	.4166	; figures corresponding, 261.
Difference,	14	
Tabular difference,	17)14(.8	

Hence, the figures corresponding to the given mantissa are 2618.

Since the characteristic is $9 - 10$, or -1, the number is a decimal with no ciphers immediately following the decimal point (Prin. 2).

Hence, the antilogarithm of $9.4180 - 10$ is .2618.

Find the antilogarithm of:

3. 0.3010.	8. 3.9545.	13. 9.3685 − 10.
4. 1.6021.	9. 0.8794.	14. 8.9932 − 10.
5. 2.9031.	10. 2.9371.	15. 8.9535 − 10.
6. 1.6669.	11. 0.8294.	16. 7.7168 − 10.
7. 2.7971.	12. 1.9039.	17. 6.7016 − 10.

340. Multiplication by logarithms.

Since logarithms are the exponents of the powers to which a constant number is to be raised, it follows that:

341. Principle. — *The logarithm of the product of two or more numbers is equal to the sum of their logarithms;* that is,

To any base, $\log (mn) = \log m + \log n.$

For, let $\log_a m = x$ and $\log_a n = y$, a being any base.

It is to be proved that $\log_a (mn) = x + y.$

§ 327, $a^x = m,$

and $a^y = n.$

Multiplying, we have $a^{x+y} = mn.$

Hence, § 328, $\log_a (mn) = x + y$

$$= \log_a m + \log_a n.$$

EXERCISES

342. 1. Multiply .0381 by 77.

Solution

Prin., § 341, $\log (.0381 \times 77) = \log .0381 + \log 77.$

$$\log .0381 = 8.5809 - 10$$
$$\log 77 = 1.8865$$
$$\text{Sum of logs} = 10.4674 - 10$$
$$= 0.4674.$$
$$0.4674 = \log 2.934.$$
$$\therefore .0381 \times 77 = 2.934.$$

NOTE. — Three figures of a number corresponding to a logarithm may be found from this table with absolute accuracy, and in most cases the fourth will be correct. In finding logarithms or antilogarithms, allowance should be made for the figures after the fourth, whenever they express .5 or more than .5 of a unit in the fourth place.

Multiply:

2. 3.8 by 56.	6. 2.26 by 85.	10. 289 by .7854.
3. 72 by 39.	7. 7.25 by 240.	11. 42.37 by .236.
4. 8.5 by 6.2.	8. 3272 by 75.	12. 2912 by .7281.
5. 1.64 by 35.	9. .892 by .805.	13. 1.414 by 2.829.

343. Division by logarithms.

Since the logarithms of two numbers to a common base represent exponents of the same number, it follows that:

344. PRINCIPLE. — *The logarithm of the quotient of two numbers is equal to the logarithm of the dividend minus the logarithm of the divisor;* that is,

To any base, $\log(m \div n) = \log m - \log n.$

For, let $\log_a m = x$ and $\log_a n = y$, a being any base.

It is to be proved that $\log_a (m \div n) = x - y.$

§ 327, $a^x = m,$

and $a^y = n.$

Dividing, we have $a^{x-y} = m \div n.$

Hence, § 328, $\log_a (m \div n) = x - y$

$= \log_a m - \log_a n.$

EXERCISES

345. 1. Divide .00468 by 73.4.

SOLUTION

Prin., § 344, $\log(.00468 \div 73.4) = \log .00468 - \log 73.4.$

$$\begin{aligned}\log .00468 &= 7.6702 - 10\\ \log 73.4 &= 1.8657\\ \hline \text{Difference of logs} &= 5.8045 - 10\\ 5.8045 - 10 &= \log .00006376.\\ \therefore .00468 \div 73.4 &= .00006376.\end{aligned}$$

2. Divide 12.4 by 16.

SOLUTION

Prin., § 344, $\log(12.4 \div 16) = \log 12.4 - \log 16.$

$$\begin{array}{l} \log 12.4 = 1.0934 = 11.0934 - 10 \\ \log 16 \quad = \qquad\quad 1.2041 \\ \hline \text{Difference of logs} = 9.8893 - 10 \end{array}$$

$$9.8893 - 10 = \log .775.$$

$$\therefore 12.4 \div 16 = .775.$$

SUGGESTION. — The positive part of the logarithm of the dividend may be made to exceed that of the divisor, if necessary to avoid subtracting a larger number from a smaller one as in the above solution, by adding 10 – 10 or 20 – 20, etc.

Divide:

3. 3025 by 55.	8. 10 by 3.14.	13. 1 by 40.
4. 4090 by 32.	9. .6911 by .7854.	14. 1 by 75.
5. 3250 by 57.	10. 2.816 by 22.5.	15. 200 by .5236.
6. .2601 by .68.	11. 4 by .00521.	16. 300 by 17.32.
7. 3950 by .250.	12. 26 by .06771.	17. .220 by .3183.

346. Extended operations in multiplication and division.

Though *negative* numbers have no common logarithms, operations involving negative numbers may be performed by considering only their *absolute values* and then giving to the result the proper sign without regard to the logarithmic work.

Since dividing by a number is equivalent to multiplying by its reciprocal, for every operation of division an operation of multiplication may be substituted. In extended operations in multiplication and division with the aid of logarithms, the latter method of dividing is the more convenient.

347. The logarithm of the reciprocal of a number is called the **cologarithm** of the number.

The cologarithm of 100 is the logarithm of $\frac{1}{100}$, which is -2.
It is written, colog $100 = -2$.

348. Since the logarithm of 1 is 0 and the logarithm of a quotient is obtained by subtracting the logarithm of the divisor from that of the dividend, it is evident that the cologarithm of a number is 0 minus the logarithm of the number, or the logarithm of the number with the sign of the logarithm changed; that is, if $\log_a m = x$, then, $\operatorname{colog}_a m = -x$.

Since subtracting a number is equivalent to adding it with its sign changed, it follows that:

349. Principle. — *Instead of subtracting the logarithm of the divisor from that of the dividend, the cologarithm of the divisor may be added to the logarithm of the dividend;* that is,

To any base, $\log(m \div n) = \log m + \operatorname{colog} n.$

EXERCISES

350. 1. Find the value of $\dfrac{.063 \times 58.5 \times 799}{458 \times 15.6 \times .029}$ by logarithms.

Solution

$$\frac{.063 \times 58.5 \times 799}{458 \times 15.6 \times .029} = .063 \times 58.5 \times 799 \times \frac{1}{458} \times \frac{1}{15.6} \times \frac{1}{.029}.$$

$$\begin{aligned}
\log .063 &= 8.7993 - 10\\
\log 58.5 &= 1.7672\\
\log 799 &= 2.9025\\
\operatorname{colog} 458 &= 7.3391 - 10\\
\operatorname{colog} 15.6 &= 8.8069 - 10\\
\operatorname{colog} .029 &= 1.5376\\
\hline
\text{log of result} &= 31.1526 - 30\\
&= 1.1526.
\end{aligned}$$

$\therefore$ result $= 14.21.$

To find Colog on slide rule: find number on D scale use other end of log scale

Find the value of:

2. $\dfrac{110 \times 3.1 \times .653}{33 \times 7.854 \times 1.7}$. .505

3. $\dfrac{15 \times .37 \times 26.16}{11 \times 8 \times .18 \times 6.67}$. .0137

4. $\dfrac{(-3.04) \times .2608}{2.046 \times .06219}$.

5. $\dfrac{600 \times 5 \times 29}{.7854 \times 25000 \times 81.7}$.

6. $\dfrac{3.516 \times 485 \times 65}{3.33 \times 17 \times 18 \times 73}$.

7. $\dfrac{.4051 \times (-12.45)}{(-8.988) \times .01442}$.

8. $\dfrac{78 \times 52 \times 1605}{338 \times 767 \times 431}$.

9. $\dfrac{.5 \times .315 \times 428}{.317 \times .973 \times 43.7}$.

351. Involution by logarithms.

Since logarithms are simply exponents, it follows that:

352. Principle. — *The logarithm of a power of a number is equal to the logarithm of the number multiplied by the index of the power;* that is,

To any base, $\log m^n = n \log m$.

For, let $\log_a m = x$, and let n be any number, a being any base.

It is to be proved that $\log_a m^n = nx$.

§ 327, $a^x = m$.

Raise each member to the nth power, Ax. 6,

$a^{nx} = m^n$. just as $a^m = r^k$ then $\log_a r^k = m$

Hence, § 328, $\log_a m^n = nx = n \log_a m$.

EXERCISES

353. 1. Find the value of $.25^2$.

Solution

Prin., § 352,

$$\log .25^2 = 2 \log .25.$$
$$\log .25 = 9.3979 - 10.$$
$$2 \log .25 = 18.7958 - 20$$
$$= 8.7958 - 10.$$
$$8.7958 - 10 = \log .06249.$$
$$\therefore .25^2 = .06249.$$

Note. — By actual multiplication it is found that $.25^2 = .0625$, whereas the result obtained by the use of the table is .06249.

Also, by multiplication, $18^2 = 324$, whereas by the use of the table it is found to be 324.1. Such inaccuracies must be expected when a four-place table is used.

Find by logarithms the value of:

2. 7^2.	7. $.78^2$.	12. 4.07^3.	17. $(\frac{3}{20})^2$.
3. 11^2.	8. 8.05^2.	13. $.543^3$.	18. $(\frac{1}{7})^3$.
4. $(-47)^2$.	9. 8.33^2.	14. $(-7)^4$.	19. $(\frac{128}{1738})^2$.
5. 4.9^2.	10. 6.61^3.	15. 1.02^5.	20. $(\frac{675}{4121})^3$.
6. 5.2^2.	11. $.714^2$.	16. 1.738^3.	21. $(\frac{1}{243})^{.4}$.

354. Evolution by logarithms.

Since logarithms are simply exponents, it follows that:

355. PRINCIPLE. — *The logarithm of a root of a number is equal to the logarithm of the number divided by the index of the required root*; that is,

To any base, $$\log \sqrt[n]{m} = \frac{\log m}{n}.$$

For, let $\log_a m = x$ and let n be any number, a being any base.

It is to be proved that $\log_a \sqrt[n]{m} = x \div n$.

§ 327, $a^x = m$.

Take the nth root of each member, Ax. 7,

$$a^{x \cdot n} = \sqrt[n]{m}.$$

Hence, § 328, $\log_a \sqrt[n]{m} = x \div n = \dfrac{\log_a m}{n}$.

EXERCISES

356. 1. Find the square root of .1296 by logarithms.

SOLUTION

Prin., § 355, $\log \sqrt{.1296} = \frac{1}{2} \log .1296$.

$\log .1296 = 9.1126 - 10$.

$$\begin{array}{r} 2)\underline{19.1126 - 20} \\ 9.5563 - 10 \end{array}$$

$9.5563 - 10 = \log .360$.

$\therefore \sqrt{.1296} = .36$.

Find by logarithms the value of:

2. $225^{\frac{1}{2}}$.
3. $12.25^{\frac{1}{2}}$.
4. $.2023^{\frac{1}{2}}$.
5. $326^{\frac{1}{2}}$.
6. $.512^{\frac{1}{3}}$.
7. $.1182^{\frac{1}{2}}$.
8. $(-1331)^{\frac{1}{3}}$.
9. $1024^{\frac{7}{10}}$.
10. $.6724^{\frac{1}{2}}$.
11. $5.929^{\frac{1}{2}}$.
12. $.4624^{\frac{1}{2}}$.
13. $1.4641^{\frac{1}{4}}$.
14. $\sqrt{2}$.
15. $\sqrt{3}$.
16. $\sqrt{5}$.
17. $\sqrt{6}$.
18. $\sqrt[3]{2}$.
19. $\sqrt[4]{4}$.
20. $\sqrt[5]{-2}$.
21. $\sqrt[3]{.027}$.
22. $\sqrt{30\frac{4}{5}}$.
23. $\sqrt{.90}$.
24. $\sqrt{.52}$.
25. $\sqrt[5]{.032}$.

Simplify the following:

26. $\dfrac{176}{15 \times 3.1416}$.
27. $\dfrac{(-100)^2}{48 \times 64 \times 11}$.
28. $\dfrac{52^2 \times 300}{12 \times .31225 \times 400000}$.
29. $\sqrt{\dfrac{400}{55 \times 3.1416}}$.
30. $50 \times \dfrac{2^{3.5}}{8^{1.63}}$.
31. $\dfrac{14.5\sqrt[3]{-1.6}}{11}$.
32. $\sqrt{\dfrac{.434 \times 96^4}{64 \times 1500}}$.
33. $\dfrac{.32 \times 5000 \times 18}{3.14 \times .1222 \times 8}$.
34. $\dfrac{11 \times 2.63 \times 4.263}{48 \times 3.263}$.
35. $\sqrt{\dfrac{3500}{(-1.06)^4}}$.

36. $2^{\frac{1}{2}} \times (\frac{1}{2})^{\frac{2}{3}} \times \sqrt[3]{\frac{3}{2}} \times \sqrt{.1}$.

37. Applying the formula $A = \pi r^2$, find the area (A) of a circle whose radius (r) is 12.35 meters. ($\pi = 3.1416$.)

38. Applying the formula $V = \frac{4}{3}\pi r^3$, find the volume (V) of a sphere whose radius (r) is 40.11 centimeters.

39. The formula $V = .7854\, d^2 l$ gives the volume of a right cylinder d units in diameter and l units long, V, d, and l being corresponding units. How many feet of No. 00 wire, which has a diameter of .3648 inches, can be made from a cubic foot of copper?

COMPLEX NUMBERS

357. Operations involving imaginary numbers are subject to the condition that

$$(\sqrt{-1})^2, \text{ or } i^2, \text{ equals } -1, \text{ not } +1.$$

358. Including all intermediate fractional and surd values the scale of real numbers may be written

$$\cdots -3 \cdots -2 \cdots -1 \cdots 0 \cdots +1 \cdots +2 \cdots +3 \cdots, \qquad (1)$$

and the scale of imaginary numbers, composed of real multiples of $+i$ and $-i$, may be written

$$\cdots -3i \cdots -2i \cdots -i \cdots 0 \cdots +i \cdots +2i \cdots +3i \cdots. \qquad (2)$$

Since the square of every real number except 0 is positive and the square of every imaginary number except $0i$, or 0, is negative, the scales (1) and (2) have no number in common except 0. Hence,

An imaginary number cannot be equal to a real number nor cancel any part of a real number.

359. The algebraic sum of a real number and an imaginary number is called a **complex number.**

$2 + 3\sqrt{-1}$, or $2 + 3i$, and $a + b\sqrt{-1}$, or $a + bi$, are complex numbers.

$a^2 + 2ab\sqrt{-1} - b^2$, or $(a^2 - b^2) + 2ab\sqrt{-1}$, is a complex number.

360. Two complex numbers that differ only in the signs of their imaginary terms are called **conjugate complex numbers.**

$a + b\sqrt{-1}$ and $a - b\sqrt{-1}$, or $a + bi$ and $a - bi$, are conjugate complex numbers.

361. Operations involving complex numbers.

EXERCISES

1. Add $3 - 2\sqrt{-1}$ and $2 + 5\sqrt{-1}$.

Solution. — Since, § 358, the imaginary terms cannot unite with real terms, the simplest form of the sum is obtained by uniting the real and the imaginary terms separately and indicating the algebraic sum of the results.

$$\begin{aligned} 3 - 2\sqrt{-1} + 2 + 5\sqrt{-1} &= (3 + 2) + (-2\sqrt{-1} + 5\sqrt{-1}) \\ &= 5 + 3\sqrt{-1}, \end{aligned}$$

Simplify the following:

2. $(5+\sqrt{-4})+(\sqrt{-9}-3)$.

3. $(2-\sqrt{-16})+(3+\sqrt{-4})$.

4. $(3-\sqrt{-8})+(4+\sqrt{-18})$.

5. $(4+\sqrt{-25})-(2+\sqrt{-4})$.

6. $(3-2\sqrt{-5})-(2-3\sqrt{-5})$.

7. $(\sqrt{-20}-\sqrt{16})+(\sqrt{-45}+\sqrt{4})$.

8. $\sqrt{-49}-2+3\sqrt{-4}-\sqrt{-1}+6$.

9. $(2-2\sqrt{-1}+3)-(\sqrt{16}-\sqrt{-16})$.

10. Expand $(a+b\sqrt{-1})(a+b\sqrt{-1})$.

Solution

$$(a+b\sqrt{-1})(a+b\sqrt{-1})=a^2+2\,ab\sqrt{-1}+(b\sqrt{-1})^2$$

§ 357, $$=a^2+2\,ab\sqrt{-1}-b^2.$$

11. Expand $(\sqrt{5}-\sqrt{-3})^2$.

Solution

$$\begin{aligned}(\sqrt{5}-\sqrt{-3})^2&=5-2\sqrt{-15}+(-3)\\&=(5-3)-2\sqrt{-15}\\&=2-2\sqrt{-15}.\end{aligned}$$

Expand:

12. $(2+3\sqrt{-1})(1+\sqrt{-1})$.

13. $(5-\sqrt{-1})(1-2\sqrt{-1})$.

14. $(\sqrt{2}+\sqrt{-2})(\sqrt{8}-\sqrt{-8})$.

15. $(2+3\,i)^2$.

16. $(2-3\,i)^2$.

17. $(a-bi)^2$.

Show that:

18. $(1+\sqrt{-3})(1+\sqrt{-3})(1+\sqrt{-3})=-8$.

19. $(-1+\sqrt{-3})(-1+\sqrt{-3})(-1+\sqrt{-3})=8$.

20. $(-\frac{1}{2}+\frac{1}{2}\sqrt{-3})(-\frac{1}{2}+\frac{1}{2}\sqrt{-3})(-\frac{1}{2}+\frac{1}{2}\sqrt{-3})=1$.

21. Divide $8+\sqrt{-1}$ by $3+2\sqrt{-1}$.

SOLUTION

$$\frac{8+\sqrt{-1}}{3+2\sqrt{-1}}=\frac{(8+\sqrt{-1})(3-2\sqrt{-1})}{(3+2\sqrt{-1})(3-2\sqrt{-1})}=\frac{26-13\sqrt{-1}}{9+4}=2-\sqrt{-1}.$$

Divide:

22. 3 by $1-\sqrt{-2}$.

23. 2 by $1+\sqrt{-1}$.

24. $4+\sqrt{4}$ by $2-\sqrt{-2}$.

25. a^2+b^2 by $a-b\sqrt{-1}$.

26. $a-bi$ by $ai+b$.

27. $(1+i)^2$ by $1-i$.

28. Find by inspection the square root of $3+2\sqrt{-10}$.

SOLUTION

$$3+2\sqrt{-10}=(5-2)+2\sqrt{5\cdot -2}=5+2\sqrt{5\cdot -2}+(-2).$$

$$\therefore\ \sqrt{3+2\sqrt{-10}}=\sqrt{5+2\sqrt{5\cdot -2}+(-2)}=\sqrt{5}+\sqrt{-2}.$$

Find by inspection the square root of:

29. $4+2\sqrt{-21}$.

30. $1+2\sqrt{-6}$.

31. $6-2\sqrt{-7}$.

32. $9+2\sqrt{-22}$.

33. $4\sqrt{-3}-1$.

34. $12\sqrt{-1}-5$.

35. $24\sqrt{-1}-7$.

36. $b^2+2\,ab\sqrt{-1}-a^2$.

37. Verify that $-1+\sqrt{-1}$ and $-1-\sqrt{-1}$ are roots of the equation $x^2+2\,x+2=0$.

362. *The sum and the product of two conjugate complex numbers are both real.*

For, let $a+b\sqrt{-1}$ and $a-b\sqrt{-1}$ be conjugate complex numbers. Their sum is $2\,a$.

Since $(\sqrt{-1})^2=-1$, their product is,

$$\begin{aligned}a^2-(b\sqrt{-1})^2&=a^2-(-b^2)\\&=a^2+b^2.\end{aligned}$$

363. *If two complex numbers are equal, their real parts are equal and also their imaginary parts.*

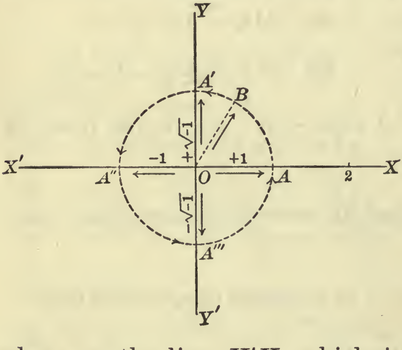
For, let $$a + b\sqrt{-1} = x + y\sqrt{-1}.$$
Then, $$a - x = (y - b)\sqrt{-1},$$
which, § 358, is impossible unless $a = x$ and $y = b$.

364. *If* $a + b\sqrt{-1} = 0$, *a and b being real, then* $a = 0$ *and* $b = 0$.

For, if $$a + b\sqrt{-1} = 0,$$
then, $$b\sqrt{-1} = - a,$$
and, squaring, we have $$- b^2 = a^2;$$
whence, $$a^2 + b^2 = 0.$$

Now, a^2 and b^2 are both positive; hence, their sum cannot be 0 unless each is separately 0; that is, $a = 0$ and $b = 0$.

365. Relation between the units $+1$, -1, $\sqrt{-1}$, and $-\sqrt{-1}$. The use of rectangular axes is a device for representing simultaneous values of two varying numbers. In the preceding discussions only real numbers have been represented. But by confining real numbers to the x-axis, it is possible, in harmony with established algebraic laws, to devote the y-axis to the representation of imaginary numbers.



In the accompanying figure the *length* of any radius of the circle represents the arithmetical unit 1. The line drawn from O to A, called the line OA, represents the positive unit $+1$, and the line OA'' represents the negative unit -1. Every real number lies somewhere on the line $X'X$, which is supposed to extend indefinitely in both directions from O. $X'X$ is called the axis of real numbers.

The *direction* of any line drawn from O, as OB, that is, the quality or *direction sign* of the number represented by that line, is determined with reference to the fixed line OA by finding what part of a revolution is required to swing the line from the position OA to the required position. By common consent revolution of the line OA is performed in a direction opposite to that of the hands of a clock, as shown by the arrows. OB is reached after $\frac{1}{6}$ of a revolution, OA' after $\frac{1}{4}$ of a revolution, OA'' after $\frac{1}{2}$ of a revolution, etc.

Since OA'', or -1, represents $\frac{1}{2}$ of a revolution of OA, the square of OA'', or $(-1)^2$, represents 1 revolution of OA, which produces OA, or $+1$. Hence, OA'', or -1, represents the square root of $+1$, or $(+1)^{\frac{1}{2}}$.

Similarly, since OA' represents $\frac{1}{2}$ of $\frac{1}{2}$ of a revolution of OA, and OA'' represents $\frac{1}{2}$ of a revolution of OA, OA' represents the square root of OA'', or of -1; that is, $OA' = \sqrt{-1}$.

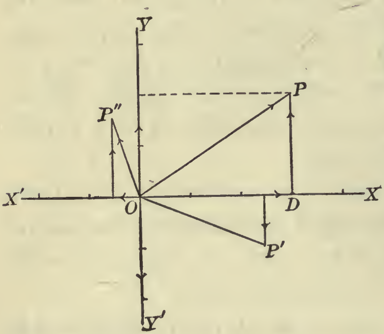
If OA'' is swung $\frac{1}{4}$ of a revolution from the position OA'' to the position OA''', OA'' will be multiplied by $\sqrt{-1}$ just as OA is multiplied by $\sqrt{-1}$ to produce OA'. Hence, the result $OA''' = -1 \cdot \sqrt{-1} = -\sqrt{-1}$.

$+1$, represented by OA, and -1, by OA'', are the units for real numbers, that is, are *real units*. Just as the real number $+a$ is represented by a line a units long extending from O toward X, and the real number $-a$ by a line a units long extending from O in the opposite direction, so the imaginary number $+a\sqrt{-1}$, or $(+\sqrt{-1}) \times a$, is represented by a line a units long extending from O toward Y, and the imaginary number $-a\sqrt{-1}$, or $(-\sqrt{-1}) \times a$, by a line a units long extending from O in the opposite direction, toward Y'. Hence, $+\sqrt{-1}$ and $-\sqrt{-1}$ are the units for imaginary numbers, that is, they are *imaginary units;* $+a\sqrt{-1}$ is called a *positive imaginary* number and $-a\sqrt{-1}$ a *negative imaginary* number.

Hence, it is seen that imaginary numbers have as much reality as real numbers. Imaginary numbers were named before their nature was understood.

366. Graphical representation of a complex number.

The sum of 3 positive real units and 2 positive imaginary units is found by counting 3 units along OX in the positive direction from O and from that point, D, measuring 2 units upward at right angles to OX in the direction of the axis of imaginary numbers. The line OP represents by its *length* and *direction* the combined effect or sum of the directed lines OD and DP, that is, the complex number $3+2\sqrt{-1}$.



The same result may be obtained by counting 2 units along OY upward from O and from the end of the second division measuring 3 units toward the right at right angles to OY in the direction of the axis of real numbers.

Hence, the line OP represents either $3+2\sqrt{-1}$ or $2\sqrt{-1}+3$. Similarly, the line OP' represents either $2\frac{1}{2}-\sqrt{-1}$ or $-\sqrt{-1}+2\frac{1}{2}$, and the line OP'' either $-\frac{1}{2}+\frac{3}{2}\sqrt{-1}$ or $\frac{3}{2}\sqrt{-1}-\frac{1}{2}$.

EXERCISES

367. Represent the following numbers graphically:

1. $3+4\sqrt{-1}$.
2. $4+2\sqrt{-1}$.
3. $3-2\sqrt{-1}$.
4. $1-3\sqrt{-1}$.
5. $2-4\sqrt{-1}$.
6. $3\sqrt{-1}-5$.
7. $-2-2\sqrt{-1}$.
8. $\frac{1}{2}-\frac{1}{2}\sqrt{-1}$.
9. $-\frac{3}{2}+\frac{3}{2}\sqrt{-1}$.

10. Represent graphically $-2+\sqrt{-13}$.

Suggestion. — The given expression may be written $-2+\sqrt{13}\sqrt{-1}$. The approximate value of $\sqrt{13}$ is 3.6.

Represent graphically:

11. $2+\sqrt{-5}$.
12. $3-\sqrt{-5}$.
13. $-4-\sqrt{-10}$.
14. $-2-\sqrt{-13}$.
15. $5+\sqrt{-17}$.
16. $2-\sqrt{-20}$.

SSARY

Abscissa. A distance measured along or parallel to the x-axis.

Absolute term. A term that does not contain an unknown number.

Absolute value. The value of a number without regard to its sign.

Addends. Numbers to be added.

Addition. The process of finding a single expression for the algebraic sum of two or more numbers.

Affected quadratic. A quadratic that contains both the second and first powers of one unknown number.

Algebra. That branch of mathematics which treats of general numbers and the nature and use of equations. It is an extension of arithmetic and it uses both figures and letters to express numbers.

Algebraic expression. A number represented by algebraic symbols.

Algebraic numbers. Positive and negative numbers, whether integers or fractions.

Algebraic sum. The result of adding two or more algebraic numbers.

Alternation. When the antecedents of a proportion are to each other as the consequents, the numbers are said to be in proportion by **alternation.**

Antecedent. The first term of a ratio.

Antilogarithm. The number that corresponds to a given logarithm.

Arithmetical progression. Same as *arithmetical series.*

Arithmetical series. A series, each term of which after the first is derived from the preceding by the addition of a constant number.

Arrangement. When a polynomial is arranged so that in passing from left to right the several powers of some letter are successively *higher* or *lower*, the polynomial is said to be **arranged** according to the **ascending** or **descending powers**, respectively, of that letter.

Axes of reference. Two straight lines that intersect, usually at right angles, used to locate a point or points in a plane.

Axiom. A principle so simple as to be self-evident.

Base of a logarithm. See *logarithm.*

Binomial. An algebraic expression of two terms.

Binomial formula. The formula by means of which any indicated power of a binomial may be expanded.

Binomial quadratic surd. A binomial surd whose surd or surds are of the second order.

Binomial surd. A binomial, one or both of whose terms are surds.

Binomial theorem. The principle by means of which any indicated power of a binomial may be expanded.

Biquadratic surd. A surd of the fourth order.

Briggs logarithms. Same as *common logarithms.*

Characteristic. The integral part of a logarithm.

Clearing an equation of fractions. The process of changing an equation containing fractions to an equation without fractions.

Coefficient. When one of the two factors into which a number can be resolved is a *known* number, it usually is written first and called the **coefficient** of the other factor.

In a broader sense, either one of the two factors into which a number can be resolved may be considered the *coefficient* of the other.

Co-factor. Same as *coefficient.*

Cologarithm. The logarithm of the reciprocal of a number is called the **cologarithm** of the number.

Common denominator. Two or more fractions that have the same denominator are said to have a **common denominator.**

Common difference. The constant number that is added to any term of an arithmetical progression to produce the next.

Common factor. A factor of each of two or more numbers.

Common logarithms. The system of logarithms whose base is 10.

Common multiple. An expression that exactly contains each of two or more given expressions.

Complete quadratic. Same as *affected quadratic.*

Complex fraction. A fraction one or both of whose terms contains a fraction.

Complex number. The algebraic sum of a real number and an imaginary number.

Composition. When the sum of the terms of one ratio of a proportion is to one of the terms as the sum of the terms of the other ratio is to its corresponding term, the numbers are said to be in proportion by **composition.**

Composition and division. When the sum of the terms of one ratio of a proportion is to their difference as the sum of the terms of the other ratio is to their difference, the numbers are said to be in proportion by **composition and division.**

Compound expression. Same as *polynomial.*

Conditional equation. An equation that is true for only certain values of its letters.

Conjugate complex numbers. Two complex numbers that differ only in the signs of their imaginary terms.

Conjugate surds. Two binomial quadratic surds that differ only in the sign of one of the terms.

Consequent. The second term of a ratio.

Consistent equations. Same as *simultaneous equations.*

Constant. A number that has the same value throughout a discussion.

Continued fraction. A complex fraction of the form $\cfrac{a}{b+\cfrac{c}{d+\cdots}}$.

Continued proportion. A multiple proportion in which each conse quent is repeated as the antecedent of the following ratio.

Coördinate axes. Same as *axes of reference.*

Coördinates. See *rectangular coördinates.*

Couplet. The two terms of a ratio.

Cube. Same as *third power.*

Cube root. One of the three equal factors of a number.

Cubic surd. A surd of the third order.

Degree of a term. The sum of the exponents of the literal factors of a rational integral term determines the **degree of the term.**

Degree of an expression. The term of highest degree in any rational integral expression determines the **degree of the expression.**

Denominator. The *divisor* in an algebraic fraction.

Dependent equations. Two or more equations that express the same relation between the unknown numbers involved are often called **dependent equations,** for each may be *derived* from any one of the others.

Derived equations. Same as *dependent equations.*

Difference. The result of subtracting one number from another. That is, the **difference** is the algebraic number that added to the subtrahend gives the minuend.

Discriminant. The expression $b^2 - 4ac$, which appears in the roots of the general quadratic equation $ax^2 + bx + c = 0$.

Dissimilar monomials. Monomials that contain different letters or the same letters with different exponents.

Dissimilar terms. Terms that contain different letters or the same letters with different exponents.

Dividend. In division, the number that is divided.

Division. The process of finding one of two factors when their product and one of the factors is given.

Division in proportion. When the difference of the terms of one ratio of a proportion is to one of the terms as the difference of the terms of the other ratio is to its corresponding term, the numbers are said to be in proportion by **division.**

Divisor. In division, the number by which the dividend is divided.

Duplicate ratio. The ratio of the squares of two numbers is called their **duplicate ratio.**

Elimination. The process of deriving from a system of simultaneous equations another system involving fewer unknown numbers.

Entire surd. A surd that has no rational coefficient except unity.

Equation. A statement of the equality of two numbers or expressions.

Equation of a problem. The statement in algebraic language of the conditions of the problem.

Equation of condition. Same as *conditional equation.*

Equation of the first degree. Same as *simple equation.*

Equation of the second degree. Same as *quadratic equation.*

Equivalent equations. Two or more equations that have the same roots, each equation having all the roots of the other.

Equivalent fractions. Fractions that are of the same value, though they may differ in form.

Even root. A root whose index is an even number.

Evolution. The process of finding any required root of a number.

Exponent. A small figure or letter placed at the right and a little above a number to indicate how many times the number is to be used as a factor.

Extraneous root. A value found for the unknown number, in the solution of an equation, that does not *satisfy* the equation.

Extremes of a proportion. The first and fourth terms.

Extremes of a series. The first and last terms.

Factor. Each of two or more numbers whose product is a given number.

Factorial n. The product of the successive integers from 1 to n or from n to 1, n being any integer.

Factoring. The process of separating a number into its factors.

Finite number. A number that cannot become either infinite or infinitesimal.

Finite series. A series consisting of a limited number of terms.

First degree equation. Same as *simple equation.*

Formula. An expression of a principle, a rule, or a law in symbols.

Fourth proportional. The fourth number of four different numbers that form a proportion.

Fourth root. One of the four equal factors of a number.

Fraction. In algebra, an indicated division ; in arithmetic, one or more of the equal parts of a unit.

Fractional equation. An equation that involves an unknown number in any denominator.

Fractional expression. An expression, any term of which is a fraction.

Fulcrum. The point or edge upon which a lever rests.

Function. An expression involving one or more letters is called a **function** of those letters.

General number. A literal number to which any value may be assigned.

Geometrical progression. Same as *geometrical series.*

Geometrical series. A series, each term of which after the first is derived by multiplying the preceding term by some constant multiplier.

Graph. A picture (line or lines) every point of which exhibits a pair of corresponding values of two related quantities.

Graph of an equation. The line or lines containing all the points, and only those, whose coördinates satisfy a given equation.

Greater than. One number is said to be **greater than** another when the remainder obtained by subtracting the second from the first is *positive.*

Higher equation. An equation that contains a higher power of the unknown number than the second.

Highest common factor. The common factor of two or more expressions that has the largest numerical coefficient and is of the highest degree.

It is equal to the product of all the common factors of the expressions.

Homogeneous equation. An equation *all* of whose terms are of the same degree with respect to the unknown numbers.

Homogeneous expression. An expression all of whose terms are of the same degree.

Identical equation. An equation whose members are identical, or such that they may be reduced to the same form.

Identity. Same as *identical equation.*

Imaginary number. A number that involves an indicated even root of a negative number.

Incomplete quadratic. Same as *pure quadratic.*

Inconsistent equations. Two or more equations that are not satisfied in common by any set of values of the unknown numbers.

Independent equations. Two or more equations that express different relations between the unknown numbers involved, and so cannot be reduced to the same equation.

Indeterminate equation. An equation that is satisfied by an unlimited number of sets of values of its unknown numbers.

Index of a power. Same as *exponent.*

Index of a root. A small figure or letter written in the opening of a radical sign to indicate what root of a number is sought.

Inequality. An algebraic expression indicating that one number is greater than or less than another.

Infinite number. A variable that may become numerically greater than any assignable number.

Infinite series. A series consisting of an unlimited number of terms.

Infinitesimal number. A variable that may become numerically less than any assignable number.

Infinity. The same as *infinite number.*

Integer. Same as *whole number.*

Integral equation. An equation that does not involve an unknown number in any denominator.

Integral expression. An expression that contains no fraction.

Inverse ratio. Same as *reciprocal ratio.*

Inversion. When the terms of each ratio of a proportion are written in *inverse* order, the numbers are said to be in proportion by **inversion.**

Involution. The process of finding any required power of an expression.

Irrational equation. An equation involving an irrational root of an unknown number.

Irrational expression. An expression containing an irrational number.

Irrational number. A number that cannot be expressed as an integer or as a fraction with integral terms.

Known number. A general number or a number whose value is known.

Less than. One number is said to be **less than** another when the remainder obtained by subtracting the second from the first is *negative.*

Lever. Any sort of a bar resting on a fixed point or edge.

Like degree. The same degree.

Like terms. Same as *similar terms.*

Limit of a variable. A constant which the value of the variable continually approaches but never reaches.

Linear equation. Same as *simple equation.*

Linear function. A function of the first degree in the variable or variables involved.

Literal coefficient. A coefficient composed of letters.

Literal equation. An equation one or more of whose known numbers is expressed by letters.

Literal numbers. Letters that are used for numbers.

Logarithm. The exponent of the power to which a fixed number, called the **base**, must be raised in order to produce a given number is called the **logarithm** of the given number.

Lowest common denominator. The denominator of lowest degree, having the least numerical coefficient, to which two or more fractions can be reduced.

It is equal to the lowest common multiple of the given denominators.

Lowest common multiple. The expression having the smallest numerical coefficient and of lowest degree that will exactly contain each of two or more given expressions.

Lowest terms. When the terms of a fraction have no common factor, the fraction is said to be in its **lowest terms.**

Mantissa. The fractional or decimal part of a logarithm.

Mean proportional. A number that serves as both means of a proportion.

Means of a proportion. The second and third terms.

Means of a series. All of the terms except the first and the last.

Members of an equation. In an equation, the number on the left of the sign of equality is called the **first member** of the equation, and the number on the right is called the **second member.**

Minuend. In subtraction, the number from which the subtraction is made.

Mixed coefficient. A coefficient composed of both figures and letters.

Mixed expression. An expression some of whose terms are integral and some fractional.

Mixed number. Same as *mixed expression.*

Mixed surd. A surd that has a rational coefficient.

Monomial. An algebraic expression of one term only.

Multiple proportion. A proportion that consists of three or more equal ratios.

Multiplicand. In multiplication, the number multiplied.

Multiplication. When the multiplier is a positive integer, the process of taking the multiplicand as many times as there are units in the multiplier.

In general, the process of finding a number that is obtained from the multiplicand just as the multiplier is obtained from unity.

Multiplier. In multiplication, the number by which the multiplicand is multiplied.

Natural numbers. The numbers 1, 2, 3, 4, and so on.

Negative number. A number less than zero.

Negative term. A term preceded by the sign $-$.

Numerator. The *dividend* in an algebraic fraction.

Numerical coefficient. A coefficient composed of figures.

Numerical equation. An equation all of whose known numbers are expressed by figures.

Odd root. A root whose index is odd.

Order of a radical or of a surd is indicated by the index of the root or by the denominator of the fractional exponent.

Ordinate. A distance measured along or parallel to the y-axis.

Origin. The intersection of the axes of reference.

Parentheses. One of the signs of aggregation ( ).

Pascal's triangle. The triangular array of coefficients of the expansion of successive powers of a binomial, beginning with the zero power.

Perfect square. An expression that may be separated into two equal factors.

Polynomial. An algebraic expression of more than one term.

Positive number. A number greater than zero.

Positive term. A term preceded by $+$, expressed or understood.

Power of a number. The product obtained when the number is used a certain number of times as a factor.

Prime factor. A factor that is a prime number.

Prime number. A number that has no factors except itself and 1.

Prime to each other. Expressions that have no common prime factor except 1 are said to be **prime to each other.**

Principal root. A real root of a number that has the same sign as the number itself.

Problem. A question that can be answered only after a course in reasoning.

Product. The result of multiplying one number by another.

Proportion. An equality of ratios.

Pure quadratic. An equation that, when simplified, contains only the second power of the unknown number.

Quadratic equation. An equation that, when simplified, contains the *square* of the unknown number, but no higher power.

Quadratic form. An expression that contains but two powers of an unknown number or expression, the exponent of one power being twice that of the other.

Quadratic formula. A formula that expresses the roots of the general quadratic equation $ax^2 + bx + c = 0$.

Quadratic function. A function of the second degree in the variable or variables involved.

Quadratic surd. A surd of the second order.

Quotient. The result of dividing one number by another.

Radical. An indicated root of a number.

Radical equation. Same as *irrational equation.*

Radical expression. An expression that involves a radical in any way.

Radicand. A number whose root is required.

Ratio. The relation of two numbers that is expressed by the quotient of the first divided by the second.

Ratio of a geometrical series. The number by which any term of the series is multiplied to produce the next.

Ratio of equality. A ratio whose antecedent is *equal to* its consequent.

Ratio of greater inequality. A ratio whose antecedent is *greater than* its consequent.

Ratio of less inequality. A ratio whose antecedent is *less than* its consequent.

Rational expression. An expression that contains no irrational number.

Rational factor of a surd. A factor whose radicand is a perfect power of a degree corresponding to the order of the surd.

Rational number. A number that is, or may be, expressed as an integer or as a fraction with integral terms.

Rationalization. The process of multiplying an expression containing a surd by any number that will make the product rational.

Rationalizing factor. The factor by which a surd expression is multiplied to render the product rational.

Rationalizing the denominator. The process of reducing a fraction having an irrational denominator to an equal fraction having a rational denominator.

Real number. A number that does not involve the even root of a negative number.

Reciprocal of a number is 1 divided by the number.

Reciprocal of a fraction is the fraction *inverted* or 1 divided by the fraction.

Reciprocal ratio. The ratio of the reciprocals of two numbers is called the **reciprocal ratio** of the numbers.

Rectangular coördinates. The abscissa and ordinate of a point referred to two perpendicular axes are called the **rectangular coördinates** of the point.

Reduction. The process of changing the form of an expression without changing its value.

Remainder in subtraction. Same as *difference.*

Root of an equation. Any number that satisfies the equation.

Root of a number. When the factors of a number are all equal, one of the factors is called a **root** of the number.

Satisfied. When an equation is reduced to an identity by the substitution of certain known numbers for the unknown numbers, the equation is said to be **satisfied.**

Second degree equation. Same as *quadratic equation.*

Second power. When a number is used *twice* as a factor, the product is called the **second power** of the number.

Second root. Same as *square root.*

Series. A succession of numbers, each of which after the first is derived from the preceding number or numbers according to some fixed law.

Sign of a fraction. The sign written before the dividing line of a fraction.

Similar monomials. Monomials that contain the same letters with the same exponents.

Similar radicals. Radicals that in their simplest form are of the same order and have the same radicand.

Similar surds. Surds that in their simplest form are of the same order and have the same radicand.

Similar terms. Terms that contain the same letters with the same exponents.

Simple equation. An integral equation that involves only the first power of one unknown number in any term when similar terms have been united.

Simple expression. Same as *monomial.*

Simplest form of a radical. A radical is in its simplest form when the index of the root is as small as possible, and when the radicand is integral and contains no factor that is a perfect power of a degree corresponding to the index of the root.

Simultaneous equations. Two or more equations that are satisfied by the same set or sets of values of the unknown numbers form a **system** of **simultaneous equations.**

Solution of an equation. The process of finding the roots of an equation.

Square. Same as *second power.*

Square root. One of the two equal factors of a number.

Substitution. When a particular number takes the place of a letter, or general number, the process is called **substitution.**

Subtraction. The process of finding one of two numbers when their sum and the other number are given.

Subtraction is the *inverse* of addition.

Subtrahend. In subtraction, the number that is subtracted.

Sum. See *algebraic sum.*

Surd. The indicated root of a *rational* number that cannot be obtained exactly.

Symmetrical equation. An equation that is not affected by interchanging the unknown numbers involved.

Term. An algebraic expression whose parts are not separated by the signs + or −.

Terms of a fraction. The numerator and denominator of a fraction.

Terms of a series. The successive numbers that form the series.

Third power. When a number is used *three* times as a factor, the product is called the **third power** of the number.

Third proportional. The consequent of the second ratio when the means of a proportion are identical.

Third root. Same as *cube root.*

Transposition. The process of removing a term from one member of an equation to the other.

Trinomial. An algebraic expression of three terms.

Trinomial square. A trinomial that is a perfect square.

Triplicate ratio. The ratio of the cubes of two numbers is called their **triplicate ratio.**

Unknown number. A number whose value is to be found.

Unlike terms. Same as *dissimilar terms.*

Variable. A number that under the conditions imposed upon it may have a series of different values.

Vary. Same as *vary directly.*

Vary directly. One quantity or number is said to **vary directly** as another, when the two depend upon each other in such a manner that if one is changed the other is changed *in the same ratio.*

Vary inversely. One quantity or number **varies inversely** as another when it varies as the *reciprocal* of the *other.*

Vary jointly. One quantity or number **varies jointly** as two others when it varies as their product.

Whole number. A unit or an aggregate of units.

X-axis. The horizontal *axis of reference* is usually called the **x-axis.**

Y-axis. The vertical *axis of reference* is usually called the **y-axis.**

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